(a, b) only

10.1 a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarization, is given by

$$\frac{d\sigma}{d\Omega}(\bar{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\bar{\varepsilon}_0\cdot\hat{n}|^2 - \frac{1}{4}\left|\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2 - \hat{n}_0\cdot\hat{n}\right]$$

where $\hat{n}_0$, $\hat{n}$ are the directions of the incident & scattered radiation, respectively, while $\bar{\varepsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\bar{\varepsilon}_0^*\cdot\bar{\varepsilon}_0 = 1$, $\hat{n}_0\cdot\bar{\varepsilon}_0 = 0$).

If all polarizations are specified, then from §10.1(C),

$$\frac{d\sigma}{d\Omega}(\hat{n}, \bar{\varepsilon}; \hat{n}_0, \bar{\varepsilon}_0) = k^4 a^6 \left|\bar{\varepsilon}^*\cdot\bar{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\bar{\varepsilon}^*)\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2 \qquad (10.14)$$

For our two (normalized) outgoing polarizations, define

$$\bar{\varepsilon}_1 = \frac{\hat{n}\times\hat{n}_0}{|\hat{n}\times\hat{n}_0|}, \qquad \bar{\varepsilon}_2 = \hat{n}\times\bar{\varepsilon}_1 = \frac{\hat{n}\times(\hat{n}\times\hat{n}_0)}{|\hat{n}\times\hat{n}_0|}$$

Sum over outgoing polarization:

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\bar{\varepsilon}_1^*\cdot\bar{\varepsilon}_0 - \frac{1}{2}\overbrace{(\hat{n}\times\bar{\varepsilon}_1^*)}^{=\bar{\varepsilon}_2}\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2 + k^4 a^6 \left|\bar{\varepsilon}_2^*\cdot\bar{\varepsilon}_0 - \frac{1}{2}\underbrace{(\hat{n}\times\bar{\varepsilon}_2^*)}_{=-\bar{\varepsilon}_1}\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2$$

$$= k^4 a^6 \left|\bar{\varepsilon}_1\cdot\bar{\varepsilon}_0 - \frac{1}{2}\bar{\varepsilon}_2\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2 + k^4 a^6 \left|\bar{\varepsilon}_2\cdot\bar{\varepsilon}_0 + \frac{1}{2}\bar{\varepsilon}_1\cdot(\hat{n}_0\times\bar{\varepsilon}_0)\right|^2$$

10.1 a), cont'd

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\left| (\hat{n}\times\hat{n}_0)\cdot\vec{\varepsilon}_0 - \frac{1}{2}\left(\hat{n}\times(\hat{n}\times\hat{n}_0)\right)\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right|^2$$

$$+ \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\left| \left(\hat{n}\times(\hat{n}\times\hat{n}_0)\right)\cdot\vec{\varepsilon}_0 + \frac{1}{2}(\hat{n}\times\hat{n}_0)\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right|^2$$

Use

$$(\hat{n}\times\hat{n}_0)\cdot\vec{\varepsilon}_0 = \hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)$$

$$\left(\hat{n}\times(\hat{n}\times\hat{n}_0)\right)\cdot\vec{\varepsilon}_0 = (\hat{n}\times\hat{n}_0)\cdot(\vec{\varepsilon}_0\times\hat{n})$$

$$= (\hat{n}\cdot\vec{\varepsilon}_0)(\hat{n}_0\cdot\hat{n}) - \hat{n}^2\underbrace{(\hat{n}_0\cdot\vec{\varepsilon}_0)}_{=0}$$

$$= (\hat{n}\cdot\vec{\varepsilon}_0)(\hat{n}_0\cdot\hat{n})$$

$$(\hat{n}\times\hat{n}_0)\cdot(\hat{n}_0\times\vec{\varepsilon}_0) = (\hat{n}\cdot\hat{n}_0)\underbrace{(\hat{n}_0\cdot\vec{\varepsilon}_0)}_{=0} - (\hat{n}\cdot\vec{\varepsilon}_0)(\hat{n}_0^2)$$

$$= -\hat{n}\cdot\vec{\varepsilon}_0$$

$$\left(\hat{n}\times(\hat{n}\times\hat{n}_0)\right)\cdot(\hat{n}_0\times\vec{\varepsilon}_0) = \left[(\hat{n}\cdot\hat{n}_0)\hat{n} - \hat{n}_0\right]\cdot(\hat{n}_0\times\vec{\varepsilon}_0)$$

$$= (\hat{n}\cdot\hat{n}_0)\,\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0) - \underbrace{\hat{n}_0\cdot(\hat{n}_0\times\vec{\varepsilon}_0)}_{=0}$$

$$= (\hat{n}\cdot\hat{n}_0)\left[\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right]$$

10.1 a), cont'd

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\left[\left| \hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0) - \frac{1}{2}(\hat{n}\cdot\hat{n}_0)\left[\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right] \right|^2 + \left| (\hat{n}\cdot\vec{\varepsilon}_0)(\hat{n}_0\cdot\hat{n}) - \frac{1}{2}\,\hat{n}\cdot\vec{\varepsilon}_0 \right|^2 \right]$$

$$= \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\Big[\left[\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right]^2 + \frac{1}{4}(\hat{n}\cdot\hat{n}_0)^2\left[\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right]^2 - (\hat{n}\cdot\hat{n}_0)\left[\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right]^2 + (\hat{n}\cdot\vec{\varepsilon}_0)^2(\hat{n}_0\cdot\hat{n})^2 + \frac{1}{4}(\hat{n}\cdot\vec{\varepsilon}_0)^2 - (\hat{n}_0\cdot\hat{n})(\hat{n}\cdot\vec{\varepsilon}_0)^2 \Big]$$

Since $\hat{n}$, $\hat{n}_0$ are unit vectors,

$$|\hat{n}\times\hat{n}_0|^2 + |\hat{n}\cdot\hat{n}_0|^2 = \cos^2\theta + \sin^2\theta = 1$$

$$= \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\Big[-|\hat{n}\times\hat{n}_0|^2\left(\frac{1}{4}\left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 + (\hat{n}\cdot\vec{\varepsilon}_0)^2\right) + \frac{1}{4}\left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 + (\hat{n}\cdot\vec{\varepsilon}_0)^2 + \left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 + \frac{1}{4}(\hat{n}\cdot\vec{\varepsilon}_0)^2 - (\hat{n}\cdot\hat{n}_0)\left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 - (\hat{n}\cdot\hat{n}_0)(\hat{n}\cdot\vec{\varepsilon}_0)^2 \Big]$$

$$= \frac{k^4 a^6}{|\hat{n}\times\hat{n}_0|^2}\Big[-|\hat{n}\times\hat{n}_0|^2\left(\frac{1}{4}\left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 + (\hat{n}\cdot\vec{\varepsilon}_0)^2\right) + \left(\frac{5}{4} - \hat{n}\cdot\hat{n}_0\right)\left(\left(\hat{n}\cdot(\hat{n}_0\times\vec{\varepsilon}_0)\right)^2 + (\hat{n}\cdot\vec{\varepsilon}_0)^2\right) \Big]$$

10.1 a), cont'd

Now, $\{\bar{\varepsilon}_0, \hat{n}_0, \hat{n}_0 \times \bar{\varepsilon}_0\}$ form a basis of unit vectors (since $\hat{n}_0 \perp \bar{\varepsilon}_0$)

$$\Rightarrow (\hat{n}\cdot\bar{\varepsilon}_0)^2 + (\hat{n}\cdot\hat{n}_0)^2 + (\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0))^2 = \hat{n}^2 = 1$$

$$\Rightarrow (\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0))^2 + (\hat{n}\cdot\bar{\varepsilon}_0)^2 = 1 - (\hat{n}\cdot\hat{n}_0)^2$$

$$= |\hat{n}\times\hat{n}_0|^2 \quad \text{as discussed earlier}$$

$$\boxed{\begin{aligned}\Rightarrow \frac{d\sigma}{d\Omega} &= \frac{k^4 a^6}{\cancel{|\hat{n}\times\hat{n}_0|^2}}\Big[-\cancel{|\hat{n}\times\hat{n}_0|^2}\Big(\frac{1}{4}(\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0))^2 + (\hat{n}\cdot\bar{\varepsilon}_0)^2\Big) \\ &\qquad + \Big(\frac{5}{4} - \hat{n}\cdot\hat{n}_0\Big)\Big(\cancel{|\hat{n}\times\hat{n}_0|^2}\Big)\Big] \\ &= k^4 a^6\Big[\frac{5}{4} - \hat{n}\cdot\hat{n}_0 - \frac{1}{4}(\hat{n}\cdot(\hat{n}_0\cdot\bar{\varepsilon}_0))^2 - (\hat{n}\cdot\bar{\varepsilon}_0)^2\Big]\end{aligned}}$$

10.1, cont'd

b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\bar{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta \cos(2\phi) \right]$$

where $\hat{n}\cdot\hat{n}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

Unraveling the description, we write

$$\hat{n} = \hat{n}_0 \cos\theta + \bar{\varepsilon}_0 \sin\theta \cos\phi + (\hat{n}_0 \times \bar{\varepsilon}_0)\sin\theta \sin\phi$$

or,

$$\hat{n}\cdot\hat{n}_0 = \cos\theta, \quad \hat{n}\cdot\bar{\varepsilon}_0 = \sin\theta\cos\phi, \quad \hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0) = \sin\theta\sin\phi$$

Plug in:

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{5}{4} - |\bar{\varepsilon}_0\cdot\hat{n}|^2 - \frac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0)|^2 - \hat{n}_0\cdot\hat{n} \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \sin^2\theta\cos^2\phi - \frac{1}{4}\sin^2\theta\sin^2\phi - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{1}{2}\sin^2\theta(1+\cos 2\phi) - \frac{1}{8}\sin^2\theta(1-\cos 2\phi) - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos\theta - \frac{5}{8}\sin^2\theta - \frac{3}{8}\sin^2\theta\cos(2\phi) \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{5}{8}(1-\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi) \right]$$

$$= k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi) \right]$$

10.2 Electromagnetic radiation with elliptic polarization, described by the polarization vector

$$\bar{\varepsilon} = \frac{1}{\sqrt{1+r^2}}\left(\bar{\varepsilon}_+ + r e^{i\alpha}\bar{\varepsilon}_-\right)$$

is scattered by a perfectly conducting sphere of radius a. Generalize the amplitude for the scattering cross section, which applies for $r=0$ or $r=\infty$, and calculate the cross section for scattering in the long wavelength limit. Show that

$$\frac{d\sigma}{d\Omega} = k^4 a^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

Compare with problem 10.1.

Use $\bar{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\bar{\varepsilon}_1 \pm i\bar{\varepsilon}_2\right)$ $\quad \bar{\varepsilon}_1 = \hat{x}, \quad \bar{\varepsilon}_2 = \hat{y}$

Then, taking appropriate linear combinations of (10.55),

$$\bar{E}_{inc} = \sum_{\ell=1}^{\infty} i^\ell \left(\frac{2\pi(2\ell+1)}{1+r^2}\right)^{1/2}\left[\left(j_\ell(kr)\bar{X}_{\ell,+1} + \frac{1}{k}\nabla\times j_\ell(kr)\bar{X}_{\ell,+1}\right)\right.$$
$$\left. + r e^{i\alpha}\left(j_\ell(kr)\bar{X}_{\ell,-1} - \frac{1}{k}\nabla\times j_\ell(kr)\bar{X}_{\ell,-1}\right)\right]$$

$$c\bar{B}_{inc} = \sum_{\ell=1}^{\infty} i^\ell \left(\frac{2\pi(2\ell+1)}{1+r^2}\right)^{1/2}\left[\left(-\frac{i}{k}\nabla\times j_\ell(kr)\bar{X}_{\ell,+1} - i j_\ell(kr)\bar{X}_{\ell,+1}\right)\right.$$
$$\left. + r e^{i\alpha}\left(-\frac{i}{k}\nabla\times j_\ell(kr)\bar{X}_{\ell,-1} + i j_\ell(kr)\bar{X}_{\ell,-1}\right)\right]$$

using the fact that $\pm$ correspond to $\varepsilon_1 \pm i\varepsilon_2$ in (10.46) & our normalization of $1/\sqrt{2}$ changes $\sqrt{4\pi}$ to $\sqrt{2\pi}$.

10.2, cont'd

Taking similar linear combinations in (10.57),
the scattered fields are

$$\vec{E}_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^l \left(\frac{2\pi(2l+1)}{1+r^2}\right)^{1/2}\left[\left(\alpha_+(l) h_l^{(1)}(kr)\vec{X}_{l,+1} + \frac{\beta_+(l)}{k}\nabla\times h_l^{(1)}(kr)\vec{X}_{l,+1}\right)\right.$$
$$\left. + re^{i\alpha}\left(\alpha_-(l) h_l^{(1)}(kr)\vec{X}_{l,-1} - \frac{\beta_-(l)}{k}\nabla\times h_l^{(1)}(kr)\vec{X}_{l,-1}\right)\right]$$

$$c\vec{B}_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^l \left(\frac{2\pi(2l+1)}{1+r^2}\right)^{1/2}\left[\left(-\frac{i}{k}\alpha_+(l)\nabla\times h_l^{(1)}(kr)\vec{X}_{l,+1} - i\beta_+(l) h_l^{(1)}(kr)\vec{X}_{l,+1}\right)\right.$$
$$\left. + re^{i\alpha}\left(\frac{-i}{k}\alpha_-(l)\nabla\times h_l^{(1)}(kr)\vec{X}_{l,-1} + i\beta_-(l) h_l^{(1)}(kr)\vec{X}_{l,-1}\right)\right]$$

Then, applying (10.63),

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2}\frac{\cancel{4}}{2(1+r^2)}\left|\sum_{l=1}^{\infty}(2l+1)^{1/2}\left[\left(\alpha_+(l)\vec{X}_{l,+1} + i\beta_+(l)\hat{n}\times\vec{X}_{l,+1}\right)\right.\right.$$
$$\left.\left. + re^{i\alpha}\left(\alpha_-(l)\vec{X}_{l,-1} - i\beta_-(l)\hat{n}\times\vec{X}_{l,-1}\right)\right]\right|^2$$

(See problem 10.6; the result there uses same normalization on polarization as here)
In the long wavelength limit, we only keep the $l=1$ terms above.

For a perfectly conducting sphere of radius a,
we use the result above (10.71):

$$\alpha_\pm(l) = -\frac{1}{2}\beta_\pm(l) \simeq -\frac{2i}{3}(ka)^3 \quad \text{for } l=1$$

10.2, cont'd

Thus, in the long wavelength limit, we have

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2}\frac{1}{(1+r^2)}\Big|(3)^{1/2}\Big[\left(\alpha_+(1)\,\bar{X}_{1,1} + i\beta_+(1)\,\hat{n}\times\bar{X}_{1,1}\right)$$
$$+ re^{i\alpha}\left(\alpha_-(1)\,\bar{X}_{1,-1} - i\beta_-(1)\,\hat{n}\times\bar{X}_{1,-1}\right)\Big]\Big|^2$$

$$\simeq \frac{\pi}{2k^2}\frac{1}{(1+r^2)}(3)\left(\frac{2}{3}\right)^2(ka)^6\Big| -i\,\bar{X}_{1,1} - 2\,\hat{n}\times\bar{X}_{1,1}$$
$$+ re^{i\alpha}\left(-i\bar{X}_{1,-1} + 2\,\hat{n}\times\bar{X}_{1,-1}\right)\Big|^2$$

$$= \frac{\pi}{2k^2}\frac{1}{(1+r^2)}\left(\frac{4}{3}\right)(ka)^6\Big|\,\bar{X}_{1,1} - 2i\hat{n}\times\bar{X}_{1,1} + re^{i\alpha}\left(\bar{X}_{1,-1} + 2i\hat{n}\times\bar{X}_{1,-1}\right)\Big|^2$$

Now, we need explicit expressions for $\bar{X}_{1,\pm 1}$.

Recall $\bar{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\, L\, Y_{lm}$ (9.119)

hence $\bar{X}_{1,\pm1} = \frac{1}{\sqrt{2}}\, L\, Y_{1,\pm1}$

Now, $L = \frac{1}{i}\,\bar{r}\times\nabla$ (9.101)

$(\hat{r}\times\hat{\theta} = \hat{\phi},\ \ \hat{r}\times\hat{\phi} = -\hat{\theta})$

$$= i\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

& $Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$ (§3.5)

10.2, cont'd

$$\Rightarrow \bar{X}_{1,\pm1} = \mp\left(\frac{3}{16\pi}\right)^{1/2}(\pm i)\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)\sin\theta\, e^{\pm i\phi}$$

$$= \mp\left(\frac{3}{16\pi}\right)^{1/2}(\pm i)\left[\hat{\theta}(\pm i)\,e^{\pm i\phi} - \hat{\phi}\cos\theta\, e^{\pm i\phi}\right]$$

$$= {}^{+}\left(\frac{3}{16\pi}\right)^{1/2}\left[\hat{\theta}e^{\pm i\phi} \pm i\hat{\phi}\cos\theta\, e^{\pm i\phi}\right]$$

$$\hat{n}\times\bar{X}_{1,\pm1} = \hat{r}\times\bar{X}_{1,\pm1} = \left(\frac{3}{16\pi}\right)^{1/2}\left[\hat{\phi} \mp i\hat{\theta}\cos\theta\right]e^{\pm i\phi}$$

$$\Rightarrow \frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2}\frac{1}{(1+r^2)}\left(\frac{4}{3}\right)(ka)^6\left(\frac{3}{16\pi}\right)$$

$$\hookrightarrow \Big|(\hat{\theta}+i\hat{\phi}\cos\theta)e^{i\phi} - 2i(\hat{\phi}-i\hat{\theta}\cos\theta)e^{i\phi}$$

$$+ re^{i\alpha}\left((\hat{\theta}-i\hat{\phi}\cos\theta)e^{-i\phi} + 2i(\hat{\phi}+i\hat{\theta}\cos\theta)e^{-i\phi}\right)\Big|^2$$

$$= \frac{1}{8}\frac{1}{1+r^2}k^4a^6\Big|\hat{\theta}\left[e^{i\phi} - 2\cos\theta e^{i\phi} + re^{i\alpha}\left(e^{-i\phi} - 2\cos\theta e^{-i\phi}\right)\right]$$

$$+ \hat{\phi}\left[i\cos\theta e^{i\phi} - 2ie^{i\phi} + re^{i\alpha}\left(-i\cos\theta e^{-i\phi} + 2ie^{-i\phi}\right)\right]\Big|^2$$

$$= \frac{1}{8}\frac{1}{1+r^2}k^4a^6\Big|\hat{\theta}\left[1 - 2\cos\theta\right]\left[1 + re^{i\alpha}e^{-2i\phi}\right]$$

$$+ i\hat{\phi}\left[\cos\theta - 2\right]\left[+1 - re^{i\alpha}e^{-2i\phi}\right]\Big|^2$$

10.2, cont'd

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{1}{8}\frac{1}{1+r^2} k^4 a^6 \Big[(1-2\cos\theta)^2 \left|1+re^{i(\alpha-2\phi)}\right|^2 + (\cos\theta - 2)^2 \left|1-re^{i(\alpha-2\phi)}\right|^2 \Big]$$

$$= \frac{1}{8}\frac{1}{1+r^2} k^4 a^6 \Big[(1-4\cos\theta+4\cos^2\theta)(1+2r\cos(\alpha-2\phi)+r^2) + (\cos^2\theta - 4\cos\theta + 4)(1-2r\cos(\alpha-2\phi)+r^2) \Big]$$

$$= \frac{1}{8}\frac{1}{1+r^2} k^4 a^6 \Big[1 + 2r\cos(\alpha-2\phi) + r^2 - 4\cos\theta - \cancel{8r\cos\theta\cos(\alpha-2\phi)} - 4r^2\cos\theta$$
$$+ 4\cos^2\theta + 8r\cos^2\theta\cos(\alpha-2\phi) + \cancel{4r^2\cos^2\theta}\ \text{ok}$$
$$+\cos^2\theta - 2r\cos^2\theta\cos(\alpha-2\phi) + r^2\cos^2\theta$$
$$-4\cos\theta + \cancel{8r\cos\theta\cos(\alpha-2\phi)} - \cancel{4r^2\cos\theta}\ \text{ok}$$
$$+4 - 8r\cos(\alpha-2\phi) + 4r^2 \Big]$$

$$= \frac{1}{8}\frac{1}{1+r^2} k^4 a^6 \Big[5 - 6r\cos(\alpha-2\phi) + 5r^2 - 8\cos\theta - 8r^2\cos\theta + 5\cos^2\theta + 6r\cos^2\theta\cos(\alpha-2\phi) + 5r^2\cos^2\theta \Big]$$

$$= \frac{1}{8}\frac{1}{1+r^2} k^4 a^6 \Big[5(1+\cos^2\theta) - 8\cos\theta + 5r^2(1+\cos^2\theta) - 8r^2\cos\theta - 6r\sin^2\theta\cos(\alpha-2\phi) \Big]$$

$$= k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\frac{r}{1+r^2}\sin^2\theta\cos(\alpha-2\phi) \right]$$

10.2, cont'd

We can get the same result from 10.1(a). To see this:

Plug $\bar{\varepsilon}_0 = \frac{1}{\sqrt{1+r^2}}\left(\bar{\varepsilon}_+ + re^{i\alpha}\bar{\varepsilon}_-\right)$

into

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\bar{\varepsilon}_0\cdot\hat{n}|^2 - \frac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0)|^2 - \hat{n}_0\cdot\hat{n}\right]$$

Take $\hat{n}_0 = \hat{z}$, $\bar{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{x}\pm i\hat{y})$, $\hat{n}_0\times\bar{\varepsilon}_\pm = \mp i\bar{\varepsilon}_\pm$

$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

Then

$$\bar{\varepsilon}_0\cdot\hat{n} = \frac{1}{\sqrt{1+r^2}}\frac{1}{\sqrt{2}}\Big[\sin\theta\cos\phi + i\sin\theta\sin\phi + re^{i\alpha}\left(\sin\theta\cos\phi - i\sin\theta\sin\phi\right)\Big]$$

$$= \frac{\sin\theta}{\sqrt{2(1+r^2)}}\left[\cos\phi\left(1+re^{i\alpha}\right) + i\sin\phi\left(1-re^{i\alpha}\right)\right]$$

$$= \frac{\sin\theta}{\sqrt{2(1+r^2)}}\left[e^{i\phi} + re^{i\alpha}e^{-i\phi}\right]$$

$$\hat{n}_0\cdot\hat{n} = \cos\theta$$

$$\hat{n}\cdot(\hat{n}_0\times\bar{\varepsilon}_0) = \frac{1}{\sqrt{1+r^2}}\left(-i\hat{n}\cdot\bar{\varepsilon}_+ + re^{i\alpha}(+i)\,\hat{n}\cdot\bar{\varepsilon}_-\right)$$

$$= \frac{1}{\sqrt{2(1+r^2)}}\Big[(-i)\left(\sin\theta\cos\phi + i\sin\theta\sin\phi\right) + ire^{i\alpha}\left(\sin\theta\cos\phi - i\sin\theta\sin\phi\right)\Big]$$

$$= \frac{-i\sin\theta}{\sqrt{2(1+r^2)}}\left[e^{i\phi} - re^{i\alpha}e^{-i\phi}\right]$$

10.2, cont'd

Comparison to 10.1(a), cont'd

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{5}{4} - |\vec{\epsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4} |\hat{n} \cdot (\hat{n}_0 \times \vec{\epsilon}_0)|^2 - \hat{n}_0 \cdot \hat{n} \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)} \left| 1 + r e^{i(\alpha - 2\phi)} \right|^2 - \frac{1}{4} \frac{\sin^2\theta}{2(1+r^2)} \left| 1 - r e^{i(\alpha-2\phi)} \right|^2 - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos\theta - \frac{\sin^2\theta}{2(1+r^2)} \left(1 + 2r\cos(\alpha - 2\phi) + r^2 \right) - \frac{1}{4} \frac{\sin^2\theta}{2(1+r^2)} \left(1 - 2r\cos(\alpha-2\phi) + r^2 \right) \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos\theta - \frac{\sin^2\theta}{2} - \frac{1}{4}\frac{\sin^2\theta}{2} - \frac{3}{4}\frac{r}{1+r^2} \sin^2\theta \cos(\alpha - 2\phi) \right]$$

$$= k^4 a^6 \left[\frac{5}{8} + \frac{\cos^2\theta}{2} + \frac{\cos^2\theta}{8} - \cos\theta - \frac{3}{4}\frac{r}{1+r^2} \sin^2\theta \cos(\alpha - 2\phi) \right]$$

$$= k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{4}\frac{r}{1+r^2} \sin^2\theta \cos(\alpha - 2\phi) \right]$$

matching the results here.

10.3 A solid uniform sphere of radius R and conductivity σ act as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c \ll 1$. The conductivity is large enough that the skin depth δ is small compared to R.

a) Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite. (Remember that $\omega \neq 0$.)

First, note $\omega R/c \ll 1 \Rightarrow \lambda \gg R$, so we are in the long-wavelength limit, hence we are in a quasi-static regime.

Next, there are no currents outside the sphere, & for harmonic fields, both the magnetic & electric fields vanish inside a perfect conductor.

Since $\bar{J}=0$ & we're in a quasi-~~static~~ static regime, we may use a magnetic scalar potential.

$$\bar{H} = -\nabla \Phi_M, \quad \bar{B} = \mu_0 \bar{H}.$$

Put the sphere at the origin, & assume the original field is $\parallel$ z axis.

We're solving $\nabla^2 \Phi_M = 0$

$$\Rightarrow \Phi_M = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

b/c of azimuthal symmetry.

Far from the sphere, $\bar{B} = B_0 \hat{z}$

$$\Rightarrow A_0 = 0, \ A_1 = -B_0/\mu_0, \ A_\ell = 0 \text{ for } \ell > 1.$$

$$\Rightarrow \Phi_M = -\frac{B_0}{\mu_0} r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

10.3(a) cont'd

Boundary condition: on the surface of a perfect conductor, $\bar{B}\cdot\hat{n} = 0$

$$\Rightarrow \left.\frac{\partial \bar{\Phi}_M}{\partial r}\right|_{r=R} = 0$$

$$\frac{\partial \bar{\Phi}_M}{\partial r} = -\frac{B_0}{\mu_0}\cos\theta - \sum_{l=0}^{\infty}(l+1)\frac{B_l}{r^{l+2}}P_l(\cos\theta)$$

$$\text{so } \left.\frac{\partial \bar{\Phi}_M}{\partial r}\right|_{r=R} = 0 \Rightarrow -\frac{B_0}{\mu_0}\cos\theta - \sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) = 0$$

For $l=1$, $+\frac{B}{\mu_0} + 2\frac{B_1}{R^3} = 0 \Rightarrow B_1 = -\frac{R^3}{2\mu_0}B$

For $l \neq 1$, $B_l = 0$

using orthogonality of Legendre polynomials.

$$\Rightarrow \bar{\Phi}_M = -\frac{B}{\mu_0}r\cos\theta - \frac{R^3}{2\mu_0}\frac{B}{r^2}\cos\theta$$

$$\bar{B} = -\mu_0\nabla\bar{\Phi}_M$$

$$= \hat{r}B\left(1-\frac{R^3}{r^3}\right)\cos\theta + \frac{\hat{\theta}}{r}B\left(r+\frac{R^3}{2r^2}\right)(-)\sin\theta$$

$$\boxed{= \hat{r}B\left(1-\left(\frac{R}{r}\right)^3\right)\cos\theta - \hat{\theta}B\left(1+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta}$$

10.3, cont'd

b) Use the technique of § 8.1 to determine the absorption cross section of the sphere. Show that it varies as $\omega^{1/2}$ provided σ is independent of frequency.

$$\frac{dP_{loss}}{da} = \frac{\mu_c \omega \delta}{4} |\vec{H}_{\parallel}|^2 \qquad (8.12)$$

$$= \frac{\mu_c \omega \delta}{4\mu_0^2} |\vec{B}_{\parallel}|^2$$

$$= \frac{\mu_c \omega \delta}{4\mu_0^2} |\vec{B}\cdot\hat{\theta}|^2$$

$$= \frac{\mu_c \omega \delta}{4\mu_0^2} \left| B\left(\frac{3}{2}\right) \sin\theta \right|^2 \qquad \text{from (a)}$$

$$= \frac{9}{16} \frac{\mu_c \omega \delta}{\mu_0^2} B^2 \sin^2\theta$$

$$P_{loss} = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi\, R^2 \frac{dP_{loss}}{da}$$

$$= 2\pi R^2 \left(\frac{9}{16}\right) \frac{\mu_c \omega \delta}{\mu_0^2} B^2 \underbrace{\int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)}_{\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{-1}^{1} = (2)\left(\frac{2}{3}\right)}$$

$$= 2\pi R^2 \left(\frac{9}{16}\right)\left(\frac{4}{3}\right) \frac{\mu_c \omega \delta}{\mu_0^2} B^2$$

$$= \frac{3\pi}{2} R^2 \frac{\mu_c \omega \delta}{\mu_0^2} B^2$$

10.3 b), cont'd

$$\sigma_{loss} = \frac{P_{loss}}{I_0}$$

for I_0 = incident flux

$$= \frac{1}{2Z_0}|E|^2 = \frac{c^2}{2Z_0}|B|^2 = \frac{1}{2\sqrt{\mu_0^3\varepsilon_0}}B^2 \quad \text{where } Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\Rightarrow \sigma_{loss} = \frac{3\pi}{\cancel{2}}R^2\frac{\mu_c\omega\delta}{\mu_0^2}\cancel{B^2}\left[\frac{1}{\cancel{2}}\frac{1}{\sqrt{\mu_0^3\varepsilon_0}}\cancel{B^2}\right]^{-1}$$

$$= 3\pi R^2\frac{\mu_c\omega\delta}{Z_0}$$

Use $\delta = \left(\frac{2}{\mu_0\sigma\omega}\right)^{1/2}$

$$\Rightarrow \boxed{\sigma_{loss} = 3\pi R^2\left(\frac{2\varepsilon_0}{\sigma}\right)^{1/2}\frac{\mu_c}{\mu_0}\omega^{1/2} \propto \omega^{1/2},}$$

as ~~[illegible]~~ claimed, so long as σ is independent of ω.