1.14 Consider the electrostatic Green functions for Dirichlet and Neumann boundary conditions on the surface ∂V bounding the volume V. Apply Green's theorem with integration variable $\bar{y}$ and $\phi = G(\bar{x}, \bar{y})$, $\psi = G(\bar{x}', \bar{y})$, with $\nabla_y^2 G(\bar{x}, \bar{y}) = -4\pi\, \delta^3(\bar{y} - \bar{x})$.
Find an expression for the difference

$$G(\bar{x}, \bar{x}') - G(\bar{x}', \bar{x})$$

in terms of an integral over the boundary surface ∂V.

$$\phi = G(\bar{x}, \bar{y}), \quad \psi = G(\bar{x}', \bar{y}),$$

$$\int_V d^3y \left[G(\bar{x}, \bar{y}) \nabla_y^2 G(\bar{x}', \bar{y}) - G(\bar{x}', \bar{y}) \nabla_y^2 G(\bar{x}, \bar{y}) \right]$$

$$= \int_{\partial V} \left[G(\bar{x}, \bar{y})\, \partial_n G(\bar{x}', \bar{y}) - G(\bar{x}', \bar{y})\, \partial_n G(\bar{x}, \bar{y}) \right] da_y$$

$LHS = -4\pi \left(G(\bar{x}, \bar{x}') - G(\bar{x}', \bar{x}) \right)$ since $\nabla_y^2 G(x, y) = -4\pi\, \delta^3(\bar{y} - \bar{x})$.

$$\Rightarrow \boxed{G(\bar{x}, \bar{x}') - G(\bar{x}', \bar{x}) = -\frac{1}{4\pi} \int_{\partial V} \left[G(\bar{x}, \bar{y})\, \partial_n G(\bar{x}', \bar{y}) - G(\bar{x}', \bar{y})\, \partial_n G(\bar{x}, \bar{y}) \right] da_y}$$

1.14, cont'd

a) For Dirichlet boundary conditions on the potential & the associated boundary conditions on the Green function, show that $G_D(\bar{x}, \bar{x}')$ must be symmetric in $\bar{x}, \bar{x}'$.

For Dirichlet boundary conditions,
recall $G_D(x, \bar{y}) = 0$ for $\bar{y}$ on ∂V.

Plugging into the equation derived, we find

$$\boxed{G_D(\bar{x}, \bar{x}') - G_D(\bar{x}', \bar{x}) = 0}$$

1.14, cont'd

b) For Neumann boundary conditions, use the boundary condition for $G_N(\bar{x}, \bar{x}')$ to show that $G_N(\bar{x}, \bar{x}')$ is not symmetric in general, but that $G_N(\bar{x}, \bar{x}') - F(\bar{x})$ is symmetric in $\bar{x}$ and $\bar{x}'$, where

$$F(\bar{x}) = \frac{1}{S}\int_{\partial V} G_N(\bar{x}, \bar{y})\, da_y$$

Neumann boundary condition: $\frac{\partial G_N}{\partial n'}(\bar{x}, \bar{x}') = -\frac{4\pi}{S}$ for $\bar{x}'$ on ∂V, $S =$ area of ∂V.

Here,

$$G(\bar{x}, \bar{x}') - G(\bar{x}', \bar{x})$$

$$= -\frac{1}{4\pi}\int_{\partial V}\left[G(\bar{x}, \bar{y})\left(-\frac{4\pi}{S}\right) - G(\bar{x}', y)\left(-\frac{4\pi}{S}\right)\right] da_y$$

$\boxed{\neq 0}$

$$= +\frac{1}{S}\int_{\partial V}\left[G(\bar{x}, \bar{y}) - G(\bar{x}', \bar{y})\right] da_y$$

$$= F(\bar{x}) - F(\bar{x}')$$

$$\Rightarrow \boxed{G(\bar{x}, \bar{x}') - F(\bar{x}) = G(\bar{x}', \bar{x}) - F(\bar{x}')}$$

Note that since $\frac{\partial}{\partial n'} F(\bar{x}) = 0$ trivially (as independent of $\bar{x}'$),

$\tilde{G}_N(\bar{x}, \bar{x}') \equiv G_N(\bar{x}, \bar{x}') - F(\bar{x})$ satisfies same

boundary conditions as $G_N(\bar{x}, \bar{x}')$.

1.14, cont'd

c) Show that the addition of $F(\bar{x})$ does not affect the potential Φ.

First, as we noted at the end of (b),
the Green's functions $G_N(\bar{x}, \bar{x}')$ and $\tilde{G}_N(\bar{x}, \bar{x}')$, for

$$\tilde{G}_N(\bar{x}, \bar{x}') \equiv G_N(\bar{x}, \bar{x}') - F(\bar{x})$$

satisfy the same boundary conditions, and so both can be used to define a Φ for the same boundary conditions.

Second, as described in § 1.9, sol'ns for Φ with specified boundary conditions are <u>unique</u>,
hence, Φ cannot depend on whether we used G_N or $\tilde{G}_N$ to compute it.

We can also see this directly.
Let Φ be the potential generated by G,
& $\tilde{\Phi}$ " " " " $\tilde{G}$.

$$\Phi(\bar{x}) = \langle \Phi \rangle_{\partial V} + \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \rho(\bar{x}') G(\bar{x}, \bar{x}') + \frac{1}{4\pi}\int_{\partial V} G(\bar{x}, \bar{x}') \frac{\partial \Phi}{\partial n'} da'$$

so

$$\tilde{\Phi}(\bar{x}) - \Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \rho(\bar{x}') (-F(\bar{x})) + \frac{1}{4\pi}\int_{\partial V} (-F(\bar{x})) \frac{\partial \Phi}{\partial n'} da'$$

$$= (-F(\bar{x})) \frac{1}{4\pi}\left[\frac{1}{\varepsilon_0}\int_V d^3x' \rho(\bar{x}') + \int_{\partial V} \frac{\partial \Phi}{\partial n'} da'\right]$$

However, $\rho = -\varepsilon_0 \nabla^2 \Phi$,
hence

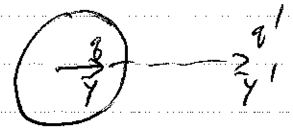
$$\frac{1}{\varepsilon_0}\int_V d^3x' \rho(\bar{x}') = -\int_V d^3x' \nabla^2 \Phi = -\int_{\partial V} \frac{\partial \Phi}{\partial n'} da'$$

& so $\underline{\tilde{\Phi}(\bar{x}) - \Phi(\bar{x}) = 0}$.

(a, b, c) only

2.2 Using the method of images, discuss the problem of a ~~point~~ point charge q <u>inside</u> a hollow, grounded, conducting sphere of radius a. Find:

a) the potential inside the sphere



Let q' be the image charge, at position y'. From cylindrical symmetry, it must be on the axis through $\bar{y}$ & the center of the sphere.

Put the coordinate origin at the center of the sphere, for simplicity, so that $\bar{y}' \parallel \bar{y}$.

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\bar{x}-\bar{y}|} + \frac{q'}{|\bar{x}-\bar{y}'|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right] \qquad \text{for } \hat{n} = \frac{\bar{x}}{|\bar{x}|}, \quad \hat{n}' = \frac{\bar{y}}{|\bar{y}|}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{x|\hat{n} - \frac{y}{x}\hat{n}'|} + \frac{q'}{y'|\hat{n}' - \frac{x}{y'}\hat{n}|}\right]$$

$$\Phi(|\bar{x}|=a) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{a|\hat{n} - \frac{y}{a}\hat{n}'|} + \frac{q'}{y'|\hat{n}' - \frac{a}{y'}\hat{n}|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{a|\hat{n} - \frac{y}{a}\hat{n}'|} + \frac{q'}{y'|\hat{n} - \frac{a}{y'}\hat{n}'|}\right]$$

Demand $= 0$,

We can satisfy this constraint if we take

$$\frac{q}{a} = -\frac{q'}{y'}, \quad \frac{y}{a} = \frac{a}{y'} \Rightarrow y' = \frac{a^2}{y}, \quad q' = -\frac{q}{a}y' = -q\frac{a}{y}$$

2.2 a), cont'd

As a consistency check, note that since $y < a$, $y' = \frac{a^2}{y} > a$,
so the image charge lies outside the sphere,
as needed.

Also: as $y \to a$, $y' \to a$ & $q' \to -q$

as $y \to 0$, $y' \to \infty$ & $q' \to 0$
- In this limit, q at the center of the sphere,
so the sphere is automatically an equipotential,
without an image charge.

$$\boxed{\Phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\bar{x}-\bar{y}|} - \frac{a/y}{|\bar{x} - \frac{a^2}{y^2}\bar{y}|}\right]}$$

where $\bar{y}$ is the location of the point charge q

2.2, cont'd

b) the induced surface-charge density

$E\cdot\hat{n} = \sigma/\varepsilon_0$ but here $\hat{n} = -\hat{r}$ since points inward.

$$\Rightarrow \sigma = -\varepsilon_0 E_r$$

$$= +\varepsilon_0 \frac{\partial \Phi}{\partial x}\bigg|_{x=a}$$

$$= +\varepsilon_0 \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[(x^2+y^2-2xy\cos\gamma)^{-1/2} - \frac{a}{y}\left(x^2+\left(\frac{a^2}{y}\right)^2 - 2x\frac{a^2}{y}\cos\gamma\right)^{-1/2}\right]\Bigg|_{x=a}$$

$$= \frac{q}{4\pi}\left[-\frac{1}{2}(x^2+y^2-2xy\cos\gamma)^{-3/2}(2x-2y\cos\gamma) - \frac{a}{y}\left(-\frac{1}{2}\right)\left(x^2+\left(\frac{a^2}{y}\right)^2-2x\frac{a^2}{y}\cos\gamma\right)^{-3/2}\left(2x-2\frac{a^2}{y}\cos\gamma\right)\right]\Bigg|_{x=a}$$

$$= \frac{q}{4\pi}\left[-(a^2+y^2-2ay\cos\gamma)^{-3/2}(a-y\cos\gamma) + \frac{a}{y}\left(a^2+\frac{a^4}{y^2}-2\frac{a^3}{y}\cos\gamma\right)^{-3/2}\left(a-\frac{a^2}{y}\cos\gamma\right)\right]$$

$$= \frac{q}{4\pi}\left[-a^{-3}\left(1+\frac{y^2}{a^2}-2\frac{y}{a}\cos\gamma\right)^{-3/2}(a-y\cos\gamma) + \frac{a}{y}\left(\frac{y^2}{a^4}\right)^{+3/2}\left(\frac{y^2}{a^2}+1-2\frac{y}{a}\cos\gamma\right)^{-3/2}\left(a-\frac{a^2}{y}\cos\gamma\right)\right]$$

$$= \frac{q}{4\pi}\frac{1}{a^3}\left(1+\frac{y^2}{a^2}-2\frac{y}{a}\cos\gamma\right)^{-3/2}\left[-(a-y\cos\gamma)+\frac{y^2}{a^2}\left(a-\frac{a^2}{y}\cos\gamma\right)\right]$$

$$= \frac{q}{4\pi}\frac{1}{a^3}\left(1+\frac{y^2}{a^2}-2\frac{y}{a}\cos\gamma\right)^{-3/2}\left[-a+\cancel{y\cos\gamma}+\frac{y^2}{a}-\cancel{y\cos\gamma}\right]$$

2.2 b), cont'd

$$\boxed{\sigma = -\frac{q}{4\pi a^2}\left(1+\frac{y^2}{a^2}-2\frac{y}{a}\cos\gamma\right)^{-3/2}\left(1-\frac{y^2}{a^2}\right)}$$

2.2 cont'd

c) the magnitude and direction of the force acting on q

$\bar{F} = q\,\bar{E}$

for $\bar{E}$ generated by the image charge.

$\bar{E}(\bar{x}) = \frac{q'}{4\pi\varepsilon_0} \frac{\bar{x} - \bar{y}'}{|\bar{x} - \bar{y}'|^3}$ for $\bar{y}'$ the location of the image charge q'

hence the force on q is

$\bar{F}(\bar{y}) = \frac{q q'}{4\pi\varepsilon_0} \frac{\bar{y} - \bar{y}'}{|\bar{y} - \bar{y}'|}$ where $\bar{y}$ is the location of q

Now, $q' = -q\frac{a}{y}$ and $\bar{y}' = \frac{a^2}{y^2}\bar{y}$

hence

$$\boxed{\bar{F} = -\frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{y} \frac{\bar{y} - \frac{a^2}{y^2}\bar{y}}{|\bar{y} - \frac{a^2}{y^2}\bar{y}|^3}} = +\frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{y} \frac{\left(\frac{a^2}{y^2} - 1\right)}{\left(\frac{a^2}{y^2} - 1\right)^3} \frac{\bar{y}}{|\bar{y}|^3}$$

$$= +\frac{1}{4\pi\varepsilon_0} q^2 \frac{a}{y}\left(1 - \frac{a^2}{y^2}\right)^{-2} \frac{\bar{y}}{|\bar{y}|^3}$$

So, the force is radially outwards, towards the enclosing sphere, of magnitude

$$\boxed{\frac{1}{4\pi\varepsilon_0} q^2 \frac{a}{y}\left(\frac{y^2}{a^2 - y^2}\right)^2 \frac{1}{y^2} = \frac{q^2}{4\pi\varepsilon_0} \frac{a y}{(a^2 - y^2)^2}}$$

(a, c, d) only

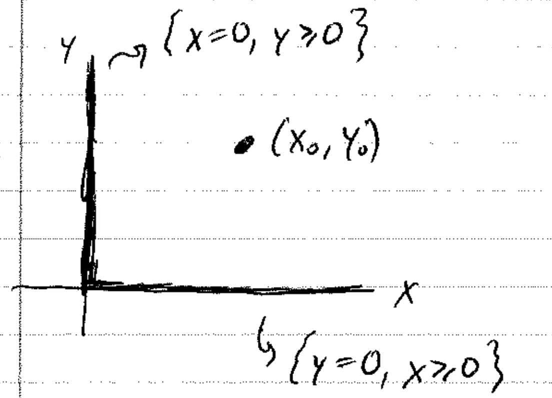
2.3 A straight-line charge with constant linear charge density λ is located perpendicular to the xy plane in the first quadrant at (x_0, y_0). The intersecting planes $x=0,\ y \geq 0$, and $y=0,\ x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

a) The well-known potential for an isolated line charge at (x_0, y_0) is

$$\Phi(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{R^2}{r^2}\right),$$

where $r^2 = (x-x_0)^2 + (y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.

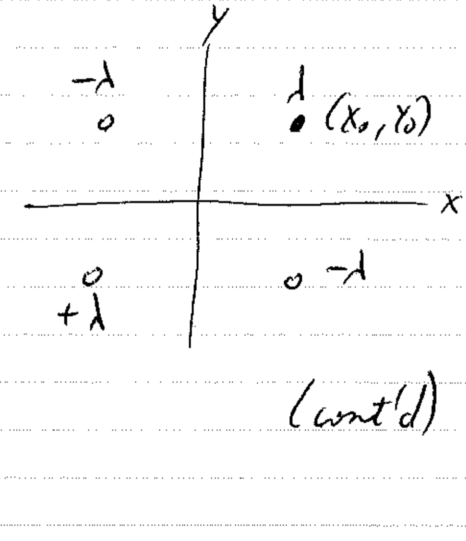
Cross-section



Shown to the left is a cross-sectional view from above of the line and the two conducting planes.

We will use the method of images. There are 3 images:

$-\lambda$ at $(-x_0, +y_0)$
$-\lambda$ at $(+x_0, -y_0)$
$+\lambda$ at $(-x_0, -y_0)$



(cont'd)

2.3 a), cont'd

The potential from the original line charge and its 3 images is

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right) - \ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right) + \ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right)\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{[(x+x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}\right)$$

Note

$$\Phi(x=0) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{[x_0^2+(y-y_0)^2][x_0^2+(y+y_0)^2]}{[x_0^2+(y-y_0)^2][x_0^2+(y+y_0)^2]}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln(1) = 0 \checkmark$$

$$\Phi(y=0) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{[(x+x_0)^2+y_0^2][(x-x_0)^2+y_0^2]}{[(x-x_0)^2+y_0^2][(x+x_0)^2+y_0^2]}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln(1) = 0 \checkmark$$

(cont'd)

2.3 a), cont'd

It remains to verify that the tangential component of $\bar{E}$ vanish along the boundary.

$$\frac{\partial \Phi}{\partial x} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} + \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\right]$$

$$E_x\Big|_{y=0} = -\frac{\partial \Phi}{\partial x}\Big|_{y=0}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\left[\cancel{\frac{2(x+x_0)}{(x+x_0)^2+y_0^2}} + \cancel{\frac{2(x-x_0)}{(x-x_0)^2+y_0^2}} - \cancel{\frac{2(x-x_0)}{(x-x_0)^2+y_0^2}} - \cancel{\frac{2(x+x_0)}{(x+x_0)^2+y_0^2}}\right]$$

$$= 0 \checkmark$$

Similarly, one can show $E_y\Big|_{x=0} = 0.$

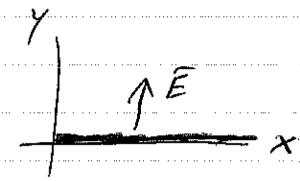
2.3, cont'd

c) Show that the total charge (per unit length in z) on the plane $\{y=0, x \geq 0\}$ is

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

What is the total charge on the plane $x=0$?

First, find the surface charge density.



$\sigma = +\varepsilon_0 \bar{E}\cdot\hat{n}$

Along the plane $\{y=0, x \geq 0\}$, $\bar{E}\cdot\hat{n} = E_y$

$$\sigma = +\varepsilon_0 E_y = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial y}\right|_{y=0}$$

$$= -\varepsilon_0 \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} + \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}\right]\Bigg|_{y=0}$$

$$= -\frac{\lambda}{4\pi}\left[\frac{-2y_0}{(x+x_0)^2+y_0^2} + \frac{2y_0}{(x-x_0)^2+y_0^2} - \frac{(-2y_0)}{(x-x_0)^2+y_0^2} - \frac{2y_0}{(x+x_0)^2+y_0^2}\right)$$

$$= -\frac{\lambda}{4\pi}\left(\frac{4y_0}{(x-x_0)^2+y_0^2} - \frac{4y_0}{(x+x_0)^2+y_0^2}\right]$$

2.3 c), cont'd

$$
\boxed{
\begin{aligned}
Q_X &= \int_0^\infty dx\ \sigma \\
&= -\frac{\lambda y_0}{\pi}\int_0^\infty dx\ \frac{1}{(x-x_0)^2+y_0^2} + \frac{\lambda y_0}{\pi}\int_0^\infty dx\ \frac{1}{(x+x_0)^2+y_0^2} \\
&= -\frac{\lambda y_0}{\pi}\int_{-x_0}^\infty \frac{1}{x^2+y_0^2}\,dx + \frac{\lambda y_0}{\pi}\int_{+x_0}^\infty \frac{1}{x^2+y_0^2}\,dx \\
&= -\frac{\lambda y_0}{\pi}\frac{1}{y_0}\tan^{-1}\left(\frac{x}{y_0}\right)\Bigg|_{-x_0}^\infty + \frac{\lambda y_0}{\pi}\frac{1}{y_0}\tan^{-1}\left(\frac{x}{y_0}\right)\Bigg|_{+x_0}^\infty \\
&= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(-\frac{x_0}{y_0}\right)\right] + \frac{\lambda}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(+\frac{x_0}{y_0}\right)\right] \\
&= -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)
\end{aligned}
}
$$

The total charge on the plane $\{x=0,\ y \geq 0\}$ should be obtained by a closely analogous computation, exchanging $x \leftrightarrow y$, hence

$$\boxed{Q_y = -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{y_0}{x_0}\right)}$$

2.3 d) Show that far from the origin ($\rho \gg \rho_0$ for $\rho^2 = x^2+y^2$ and $\rho_0^2 = x_0^2+y_0^2$), the leading term in the potential is

$$\Phi \to \Phi_{asym} = \frac{4\lambda}{\pi\epsilon_0}\,\frac{(x_0 y_0)(xy)}{\rho^4}.$$

Interpret.

Write

$$\Phi(x,y) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{[\rho^2+2xx_0-2yy_0+\rho_0^2][\rho^2-2xx_0+2yy_0+\rho_0^2]}{[\rho^2-2xx_0-2yy_0+\rho_0^2][\rho^2+2xx_0+2yy_0+\rho_0^2]}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\left[1+2\frac{(xx_0-yy_0)}{\rho^2}+\frac{\rho_0^2}{\rho^2}\right]\left[1+2\frac{(-xx_0+yy_0)}{\rho^2}+\frac{\rho_0^2}{\rho^2}\right]\right.$$
$$\left.\left[1+2\frac{(-xx_0-yy_0)}{\rho^2}+\frac{\rho_0^2}{\rho^2}\right]^{-1}\left[1+2\frac{(xx_0+yy_0)}{\rho^2}+\frac{\rho_0^2}{\rho^2}\right]^{-1}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\left[1+\frac{2\rho_0^2}{\rho^2}-4\frac{(xx_0-yy_0)^2}{\rho^4}+\frac{\rho_0^4}{\rho^4}\right]\right.$$
$$\left.\left[1+2\frac{\rho_0^2}{\rho^2}-4\frac{(xx_0+yy_0)^2}{\rho^4}+\frac{\rho_0^4}{\rho^4}\right]^{-1}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\left[1+\frac{2\rho_0^2}{\rho^2}-4\frac{(xx_0-yy_0)^2}{\rho^4}+\frac{\rho_0^4}{\rho^4}\right]\right.$$
$$\left.\left[1-2\frac{\rho_0^2}{\rho^2}+4\frac{(xx_0+yy_0)^2}{\rho^4}-\frac{\rho_0^4}{\rho^4}+\left(\frac{2\rho_0^2}{\rho^2}\right)^2+O(\rho^{-6})\right]\right)$$

(cont'd)

2.3 d), cont'd

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\left[1 + 2\frac{\rho_0^2}{\rho^2} - 4\frac{(xx_0 - yy_0)^2}{\rho^4} + \frac{\rho_0^4}{\rho^4}\right]\cdot\left[1 - 2\frac{\rho_0^2}{\rho^2} + 4\frac{(xx_0+yy_0)^2}{\rho^4} + 3\frac{\rho_0^4}{\rho^4} + O(\rho^{-6})\right]\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(1 + 4\,\frac{(xx_0+yy_0)^2 - (xx_0-yy_0)^2}{\rho^4} + \cancel{4\frac{\rho_0^4}{\rho^4}} - \cancel{4\left(\frac{\rho_0^2}{\rho^2}\right)^2} + O(\rho^{-6})\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(1 + \frac{4\left[\cancel{(xx_0)^2} + \cancel{(yy_0)^2} + 2(xx_0)(yy_0) - \cancel{(xx_0)^2} - \cancel{(yy_0)^2} + 2(xx_0)(yy_0)\right]}{\rho^4} + O(\rho^{-6})\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(1 + \frac{16(xx_0)(yy_0)}{\rho^4} + O(\rho^{-6})\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\,\frac{16(xx_0)(yy_0)}{\rho^4} + O(\rho^{-6})$$

$$\boxed{= \frac{4\lambda}{\pi\varepsilon_0}\,\frac{(xx_0)(yy_0)}{\rho^4} + O(\rho^{-6})}$$

This is essentially the potential from a two-dimensional quadrupole moment.