11.3 Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with velocity

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

This is an alternative way to derive the parallel-velocity addition law.

Under the first Lorentz transformation, for a velocity $\parallel x_1$,

$$x_0' = \gamma_1 (x_0 - \beta_1 x_1) \qquad (11.16)$$

$$x_1' = \gamma_1 (x_1 - \beta_1 x_0)$$

$$x_2' = x_2, \quad x_3' = x_3$$

for $\beta_1 = v_1/c, \quad \gamma_1 = (1 - v_1^2/c^2)^{-1/2}$

After the second Lorentz transformation,

$$x_0'' = \gamma_2 (x_0' - \beta_2 x_1')$$

$$x_1'' = \gamma_2 (x_1' - \beta_2 x_0')$$

$$x_2'' = x_2', \quad x_3'' = x_3'$$

Simplify:

$$x_0'' = \gamma_2 \left[\gamma_1 (x_0 - \beta_1 x_1) - \beta_2 \gamma_1 (x_1 - \beta_1 x_0) \right]$$

$$= x_0 (\gamma_2 \gamma_1 + \gamma_1 \gamma_2 \beta_1 \beta_2) - x_1 (\gamma_1 \gamma_2 \beta_1 + \gamma_1 \gamma_2 \beta_2)$$

$$= (\gamma_1 \gamma_2) \left[(1 + \beta_1 \beta_2) x_0 - (\beta_1 + \beta_2) x_1 \right]$$

$$x_1'' = \gamma_2 \left[\gamma_1 (x_1 - \beta_1 x_0) - \beta_2 \gamma_1 (x_0 - \beta_1 x_1) \right]$$

$$= \gamma_1 \gamma_2 \left[(1 + \beta_1 \beta_2) x_1 - (\beta_1 + \beta_2) x_0 \right)$$

11.3, cont'd

Define $\gamma = \gamma_1\gamma_2(1+\beta_1\beta_2)$, $\quad \beta\gamma = \gamma_1\gamma_2(\beta_1+\beta_2)$

Write $\gamma = (1-v^2/c^2)^{-1/2}$, $\beta = v/c$, & solve for v:

$$\beta\gamma = \gamma_1\gamma_2(\beta_1+\beta_2)$$

$$\Rightarrow \beta^2\gamma^2 = \gamma_1^2\gamma_2^2(\beta_1+\beta_2)^2$$

$$\frac{\beta^2}{1-\beta^2} = (1-\beta_1^2)^{-1}(1-\beta_2^2)^{-1}(\beta_1+\beta_2)^2$$

$$\Rightarrow \beta^2 = (1-\beta^2)\left[(1-\beta_1^2)^{-1}(1-\beta_2^2)^{-1}(\beta_1+\beta_2)^2\right]$$

$$\Rightarrow \beta^2\left[1+(1-\beta_1^2)^{-1}(1-\beta_2^2)^{-1}(\beta_1+\beta_2)^2\right] = (1-\beta_1^2)^{-1}(1-\beta_2^2)^{-1}(\beta_1+\beta_2)^2$$

$$\beta^2 = \frac{(\beta_1+\beta_2)^2}{(1-\beta_1^2)(1-\beta_2^2)+(\beta_1+\beta_2)^2}$$

$$= \frac{(\beta_1+\beta_2)^2}{(1-\cancel{\beta_1^2}-\cancel{\beta_2^2}+\beta_1^2\beta_2^2)+(\cancel{\beta_1^2}+\cancel{\beta_2^2}+2\beta_1\beta_2)}$$

$$= \frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}$$

$$\Rightarrow \boxed{v = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}}$$

Pick + root to have correct nonrelativistic limit.

11.3, cont'd

Need to check that $\gamma = \gamma_1 \gamma_2 (1+\beta_1\beta_2)$ is satisfied.

$$\beta^2 = \left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2$$

$$\Rightarrow 1-\beta^2 = \frac{(1+\beta_1\beta_2)^2 - (\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}$$

$$= \frac{1+\beta_1^2\beta_2^2 + \cancel{2\beta_1\beta_2} - \beta_1^2 - \beta_2^2 - \cancel{2\beta_1\beta_2}}{(1+\beta_1\beta_2)^2}$$

$$\Rightarrow \gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1-\beta_1^2}\,\frac{1}{1-\beta_2^2}\,(1+\beta_1\beta_2)^2$$

which is consistent with the above. ✓

11.5 A coordinate system K' moves with a velocity $\bar{v}$ relative to another system K. In K' a particle has velocity $\bar{u}'$ & acceleration $\bar{a}'$.
Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\bar{v}$ are

$$\bar{a}_{\parallel} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3}\,\bar{a}'_{\parallel}$$

$$\bar{a}_{\perp} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3}\left(\bar{a}'_{\perp} + \frac{\bar{v}}{c^2}\times\left(\bar{a}'\times\bar{u}'\right)\right)$$

Recall $\bar{u} = c\dfrac{\partial \bar{x}}{\partial x_0}$, $\quad \bar{a} = \dfrac{\partial \bar{u}}{\partial x_0}$

& for $\bar{v} \parallel x_1$,

$$\left.\begin{aligned} x_0 &= \gamma(x_0' + \beta x_1') \\ x_1 &= \gamma(x_1' + \beta x_0') \\ x_2 &= x_2', \quad x_3 = x_3' \end{aligned}\right\} \quad (11.18)$$

$$u_1 = c\frac{dx_1}{dx_0} = c\,\frac{dx_0'}{dx_0}\,\frac{dx_1}{dx_0'}\ ;$$

$$\frac{dx_1}{dx_0'} = \gamma\left(\frac{dx_1'}{dx_0'} + \beta\right) = \gamma\left(\frac{u_1'}{c} + \beta\right)$$

$$\frac{dx_0}{dx_0'} = \gamma\left(1 + \beta\frac{dx_1'}{dx_0'}\right) = \gamma\left(1 + \beta\frac{u_1'}{c}\right)$$

$$\frac{dx_0'}{dx_0} = \left[\gamma\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}$$

11.5, cont'd

$$u_1 = c\frac{dx_0'}{dx_0}\frac{dx_1}{dx_0'} = \left[\cancel{\gamma}\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}\cancel{\gamma}\left(u_1'+c\beta\right)$$

$$= \frac{u_1'+c\beta}{1+\frac{\beta u_1'}{c}}$$

$$u_2 = c\frac{dx_0'}{dx_0}\frac{dx_2}{dx_0'} = \left[\gamma\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}\left(u_2'\right)$$

$$u_3 = \left[\gamma\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}\left(u_3'\right)$$

$$a_1 = \frac{du_1}{dx_0} = \frac{dx_0'}{dx_0}\frac{du_1}{dx_0'}$$

$$= \left[\gamma\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}\left\{\frac{a_1'}{1+\beta u_1'/c} - \frac{(\beta/c)(a_1')(u_1'+c\beta)}{(1+\beta u_1'/c)^2}\right\}$$

$$= \frac{1}{\gamma(1+\beta u_1'/c)^3}\left[a_1'\left(1+\cancel{\beta u_1'/c}\right) - a_1'\frac{\beta}{c}\left(\cancel{u_1'}+c\beta\right)\right]$$

$$= \frac{1}{\gamma(1+\beta u_1'/c)^3}\left[a_1'\underbrace{\left(1-\beta^2\right)}_{=\gamma^{-2}}\right]$$

$$= \frac{1}{\gamma^3(1+\beta u_1'/c)^3}\, a_1'$$

11.5, cont'd

$$a_2 = \frac{du_2}{dx_0} = \frac{dx_0'}{dx_0}\frac{du_2}{dx_0'}$$

$$= \left[\gamma\left(1+\frac{\beta u_1'}{c}\right)\right]^{-1}\left\{-\frac{u_2'}{\gamma(1+\beta u_1'/c)^2}\left(\frac{\beta}{c}a_1'\right) + \frac{a_2'}{\gamma(1+\beta u_1'/c)}\right\}$$

$$= \frac{1}{\gamma^2(1+\beta u_1'/c)^3}\underbrace{\left[-\frac{\beta}{c}u_2' a_1' + a_2'(1+\beta u_1'/c)\right]}_{= a_2' + \frac{\beta}{c}(u_1' a_2' - u_2' a_1')}$$

a_3 is similar.

Writing these expressions for a_1, a_2, a_3 in more covariant form, we get from a_1,

$$\boxed{\bar{a}_{\parallel} = \frac{\bar{a}_{\parallel}'}{\gamma^3\left(1+\frac{\bar{\beta}\cdot\bar{u}'}{c}\right)^3} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3}\bar{a}_{\parallel}'}$$

using $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$, $\bar{\beta} = \bar{v}/c$

$$\boxed{\bar{a}_{\perp} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3}\left[\bar{a}_{\perp}' + \frac{\bar{v}}{c^2}\times(\bar{a}'\times\bar{u}')\right]}$$

Using the fact that for $\bar{\beta} = \beta\hat{x}_1$,

$$\bar{\beta}\times(\bar{a}'\times\bar{u}') = (\bar{\beta}\cdot\bar{u}')\bar{a}' - (\bar{\beta}\cdot\bar{a}')\bar{u}'$$
$$= u_1'\bar{a}' - a_1'\bar{u}'$$