11.14 a) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$, and $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ in terms of $\bar{E}$ and $\bar{B}$. Are there any other invariants quadratic in the field strengths $\bar{E}$ and $\bar{B}$?

Recall

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}, \qquad (11.137)$$

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \qquad (11.140)$$

$$g_{\alpha\beta} = \text{diag}\,(+1, -1, -1, -1) \qquad (11.69)$$

$$F^{\alpha\beta}F_{\alpha\beta} = F^{\alpha\beta}F^{\gamma\delta}\, g_{\alpha\gamma}\, g_{\beta\delta}$$

$$= -(2)\,\bar{E}^2 + 2\,\bar{B}^2$$

$$= 2\left(|B|^2 - |E|^2\right)$$

$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -2\,\bar{E}\cdot\bar{B} - 2\,\bar{E}\cdot\bar{B}$$

$$= -4\,\bar{E}\cdot\bar{B}$$

$$\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -2\,\bar{B}^2 + 2\bar{E}^2$$

$$= 2\left(\bar{E}^2 - \bar{B}^2\right)$$

Since there are no other quadratic tensor products, there are no other quadratic invariants.

11.14, cont'd

b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other inertial frame? What are the criteria imposed on $\bar{E}, \bar{B}$ such that there is an inertial frame in which there is no electric field?

Recall

$$\bar{E}' = \gamma(\bar{E} + \bar{\beta}\times\bar{B}) - \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{E})$$
$$\bar{B}' = \gamma(\bar{B} - \bar{\beta}\times\bar{E}) - \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{B}) \qquad (11.149)$$

S'pose in the unprimed frame, $\bar{B}=0$, $\bar{E}\neq 0$

Then

$$\bar{E}' = \gamma\bar{E} - \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{E})$$
$$\bar{B}' = -\gamma\bar{\beta}\times\bar{E}$$

S'pose further that in the primed frame, $\bar{E}'=0$

$$\Rightarrow \gamma\bar{E} = \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{E})$$

$$\Rightarrow \bar{E}\parallel\bar{\beta}, \quad |\bar{E}| = \frac{\gamma}{\gamma+1}\beta^2|\bar{E}|$$

$$\Rightarrow 1 = \frac{\gamma}{\gamma+1}\beta^2 = \frac{\gamma}{\gamma+1}(1-\gamma^{-2}) = \frac{\gamma-\gamma^{-1}}{\gamma+1}$$

$\Rightarrow \gamma+1 = \gamma-\gamma^{-1} \Rightarrow 1 = -\gamma^{-1} \Rightarrow \underline{\gamma=-1}$ is the only sol'n for finite γ.

<u>not possible</u>

11.14 (b) cont'd

Alternate sol'n for first part:

Recall $F^{\alpha\beta}F_{\alpha\beta}$ is Lorentz invariant.

If $\vec{B}=0$, $F^{\alpha\beta}F_{\alpha\beta} = -2\vec{E}^2 < 0$

If $\vec{E}=0$, $F^{\alpha\beta}F_{\alpha\beta} = +2\vec{B}^2 > 0$

~~Since the~~

If there were such a Lorentz transformation, $F^{\alpha\beta}F_{\alpha\beta}$ would be invariant, but as the signs differ, no such Lorentz transformation can exist.

11.14 b), cont'd

What are the criteria to ensure that a frame has no electric field?

Solve $\quad 0 = \bar{E}' = \gamma(\bar{E} + \bar{\beta}\times\bar{B}) - \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot E)$

$$\Rightarrow \boxed{\bar{\beta}\times\bar{B} = -\bar{E} + \frac{\gamma}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{E})}$$

In terms of ~~…~~ Lorentz scalars,
~~…~~ necessary conditions are that

$F^{\alpha\beta}F_{\alpha\beta} > 0 \qquad$ since $= 2(\bar{B}^2 - E^2)$

$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = 0 \qquad$ since $= -4\bar{E}\cdot\bar{B}$

11.14, cont'd

c) For macroscopic fields $\vec{E}, \vec{B}$ form the field tensor $F^{\alpha\beta}$ and $\vec{D}, \vec{H}$ the tensor $G^{\alpha\beta}$. What further invariants can be formed? What are their explicit expressions in terms of the 3-vector fields?

$$G^{\alpha\beta} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix}$$

$$\mathcal{D}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} G_{\gamma\delta} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & D_z & -D_y \\ H_y & -D_z & 0 & D_x \\ H_z & D_y & -D_x & 0 \end{pmatrix}$$

$$G^{\alpha\beta} G_{\alpha\beta} = -2\vec{D}^2 + 2\vec{H}^2$$

$$\mathcal{D}^{\alpha\beta} \mathcal{D}_{\alpha\beta} = -2\vec{H}^2 + 2\vec{D}^2$$

$$\mathcal{D}^{\alpha\beta} G_{\alpha\beta} = -4\vec{D}\cdot\vec{H}$$

$$F^{\alpha\beta} G_{\alpha\beta} = -2\vec{E}\cdot\vec{D} + 2\vec{B}\cdot\vec{H}$$

$$\mathcal{F}^{\alpha\beta} G_{\alpha\beta} = -2\vec{B}\cdot\vec{D} - 2\vec{E}\cdot\vec{H}$$

$$F^{\alpha\beta} \mathcal{D}_{\alpha\beta} = -2\vec{E}\cdot\vec{H} - 2\vec{B}\cdot\vec{D}$$

$$\mathcal{F}^{\alpha\beta} \mathcal{D}_{\alpha\beta} = -2\vec{B}\cdot\vec{H} + 2\vec{E}\cdot\vec{D}$$

(a) only

11.18 The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by

$$E_1 = E_1' = -\frac{q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$E_2 = \gamma E_2' = \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$B_3 = \gamma\beta E_2' = \beta E_2$, other components vanishing, as in (11.152),

become more and more concentrated as $\beta \to 1$. Choose axes s.t. the charge moves along the z axis in pos' direction, passing origin at $t=0$. Let the spatial coordinates of the observation point be (x, y, z), and define the transverse vector $\bar{r}_\perp$ with components x, y. Consider fields & sources in the limit $\beta \to 1$.

a) Show that the fields can be written as

$$\bar{E} = 2q\frac{\bar{r}_\perp}{r_\perp^2}\,\delta(ct-z), \quad \bar{B} = 2q\frac{\hat{v}\times\bar{r}_\perp}{r_\perp^2}\,\delta(ct-z)$$

where $\hat{v}$ is a unit vector in the direction of the particle's velocity.

(Recall, from above (11.17), that in the text's conventions,
$x_1 = z, \quad x_2 = x, \quad x_3 = y$.)

$\bar{\beta} = \frac{v}{c}\hat{x}_1$. In the rest frame of the particle,

$$\bar{E}' = q\frac{\bar{r}'}{r'^3}, \quad \bar{B}' = 0$$

In lab-frame, from (11.149),

$$\bar{E} = \gamma\bar{E}' - \frac{\gamma^2}{\gamma+1}\bar{\beta}(\bar{\beta}\cdot\bar{E}') \quad , \quad \bar{B} = \gamma\bar{\beta}\times\bar{E}'$$

11.18 a), cont'd

$$x_1' = \gamma(x_1 - \beta x_0) = \gamma(x_1 - vt)$$

$$x_2' = x_2, \quad x_3' = x_3$$

$$\Rightarrow \bar{r}' = \bar{r}_\perp + \gamma(x_1 - vt)\hat{x}_1$$

$$\bar{E}' = \frac{q\left(\bar{r}_\perp + \gamma(x_1 - vt)\hat{x}_1\right)}{\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{3/2}}$$

$$\bar{E} = \gamma\bar{E}' - \frac{\gamma^2}{\gamma+1}\underbrace{\left(\frac{v}{c}\hat{x}_1\right)}_{\bar{\beta}}\underbrace{\frac{q\gamma(x_1 - vt)}{\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{3/2}}\left(\frac{v}{c}\right)}_{=\bar{\beta}\cdot\bar{E}}$$

$$= q\gamma\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{-3/2}\left\{\bar{r}_\perp + \gamma(x_1 - vt)\hat{x}_1 - \frac{\gamma^2}{\gamma+1}\underbrace{\beta^2}_{1-\gamma^{-2}}(x_1 - vt)\hat{x}_1\right\}$$

$$= q\gamma\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{-3/2}\left\{\bar{r}_\perp + \gamma(x_1 - vt)\hat{x}_1 - \underbrace{\frac{\gamma^2 - 1}{\gamma+1}}_{=\gamma-1}(x_1 - vt)\hat{x}_1\right\}$$

$$= q\gamma\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{-3/2}\left\{\bar{r}_\perp + (x_1 - vt)\hat{x}_1\right\}$$

We need to take the limit $\beta \to 1$ or $\gamma \to \infty$.

To that end, we need an alternative representation of the Dirac delta function, which I'll describe next.

11.18 a), cont'd

Claim $\lim_{x\to\infty} \frac{x}{(1+x^2y^2)^{3/2}} = 2\delta(y)$

Check:

• 1st, note for $y \neq 0$,

$$\lim_{x\to\infty} \frac{x}{(1+x^2y^2)^{3/2}} = \lim_{x\to\infty} \frac{x}{x^3y^3} = 0$$

but this argument breaks down for $y=0$.

• Next, claim

$$\int_{-\infty}^{\infty} dy \, \frac{x}{(1+x^2y^2)^{3/2}} = 2 \quad \text{for all } x \neq 0.$$

Write $y = \frac{1}{x}\tan\theta$, so $1+x^2y^2 = 1+\tan^2\theta = \sec^2\theta$
$dy = \frac{1}{x}\sec^2\theta \, d\theta$

$$\int_{-\infty}^{\infty} dy \, \frac{x}{(1+x^2y^2)^{3/2}} = \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{x}}\sec^2\theta \, d\theta \, \frac{\cancel{x}}{\sec^3\theta}$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_{-\pi/2}^{\pi/2} = 2$$

Since the limit vanishes for $y \neq 0$,
& the integral $= 2$ independent of x,
we conclude

$$\lim_{x\to\infty} \frac{x}{(1+x^2y^2)^{3/2}} = 2\delta(y)$$

11.18 a), cont'd

Applying this result, we see

$$\lim_{\gamma\to\infty} \frac{\gamma}{\left[r_\perp^2 + \gamma^2(x_1 - vt)^2\right]^{3/2}} = \frac{1}{r_\perp^3} \lim_{\gamma\to\infty} \frac{\gamma}{\left(1 + \gamma^2 \frac{(x_1-vt)^2}{r_\perp^2}\right)^{3/2}}$$

$$= \frac{1}{r_\perp^3}\,(2)\,\delta\left(\frac{x_1 - vt}{r_\perp}\right)$$

$$= \frac{2}{r_\perp^2}\,\delta(x_1 - vt)$$

& so

$$\boxed{\lim_{\gamma\to\infty} \bar{E} = \frac{2q}{r_\perp^2}\,\delta(x_1 - vt)\left(\bar{r}_\perp + (x_1 - vt)\hat{x}_1\right)}$$

$$\boxed{= 2q\,\frac{\bar{r}_\perp}{r_\perp^2}\,\delta(x_1 - vt)}$$

$$\boxed{= 2q\,\frac{\bar{r}_\perp}{r_\perp^2}\,\delta(x_1 - ct)}$$ since $v \to c$ as $\gamma \to \infty$

11.18 a), cont'd

$$\bar{B} = \gamma \bar{\beta} \times \bar{E}'$$

$$= q\bar{\beta} \times \frac{\gamma\left(\bar{r}_\perp + \gamma(x_1 - vt)\hat{x}_1\right)}{\left[r_\perp^2 + \gamma^2 (x_1 - vt)^2\right]^{3/2}}$$

$$\lim_{\gamma \to \infty} \bar{B} = q\bar{\beta} \times \frac{\bar{r}_\perp}{r_\perp^2}(2)\,\delta(x_1 - vt)$$

$\bar{\beta} = \hat{v}$ since $|\bar{v}| \simeq c$ for the limit

$$= 2q\,\frac{\hat{v} \times \bar{r}_\perp}{r_\perp^2}\,\delta(x_1 - vt)$$

$$\simeq 2q\,\frac{\hat{v} \times \bar{r}_\perp}{r_\perp^2}\,\delta(x_1 - ct)$$

12.2 a) Show from Hamilton's principle that Lagrangians that differ only by a total time derivative ~~of~~ are equivalent in the sense that they yield the same Euler-Lagrange equations of motion.

Suppose $L_2 = L_1 + \frac{d}{dt} f(q_i(t), t)$

$$= \cancel{L_1 + \frac{\partial f}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial f}{}}$$

$$= L_1 + \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t}$$

Then,

$$\frac{\partial L_2}{\partial q_i} = \frac{\partial L_1}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t}$$

$$\frac{\partial L_2}{\partial \dot{q}_i} = \frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_i} - \frac{\partial L_2}{\partial q_i} = \frac{d}{dt}\left[\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\right] - \left[\frac{\partial L_1}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t}\right]$$

$$= \underbrace{\left(\frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i} - \frac{\partial L_1}{\partial q_i}\right)}_{=0} + \cancel{\frac{\partial^2 f}{\partial q_i \partial q_j}}\dot{q}_j + \cancel{\frac{\partial^2 f}{\partial q_i \partial t}} - \cancel{\frac{\partial^2 f}{\partial q_i \partial q_j}}\dot{q}_j - \cancel{\frac{\partial^2 f}{\partial q_i \partial t}}$$

$$= 0$$

Hence L_1 obeys Euler-Lagrange

$\Leftrightarrow$ L_2 obeys Euler-Lagrange

12.2, cont'd

b) Show explicitly that the gauge transformation $A^\alpha \mapsto A^\alpha + \partial^\alpha \Lambda$ of the potential in the charged-particle Lagrangian

$$L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{e}{\gamma c} U_\alpha A^\alpha$$

merely generates another equivalent Lagrangian.

After gauge transformation,

$$L \mapsto L' = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{e}{\gamma c} U_\alpha A^\alpha - \frac{e}{\gamma c} U_\alpha \partial^\alpha \Lambda$$

$$= L - \frac{e}{\gamma c} U_\alpha \partial^\alpha \Lambda$$

hence

$$\delta L = -\frac{e}{\gamma c} U_\alpha \partial^\alpha \Lambda$$

Use $U = (\gamma c, \gamma \bar{u})$

$$\Rightarrow \delta L = -\frac{e}{\cancel{\gamma} c}\cancel{\gamma}\left(\cancel{c}\frac{1}{\cancel{c}}\frac{\partial}{\partial t} + \bar{u}\cdot\nabla\right)\Lambda$$

Since $\Lambda = \Lambda(\bar{x}(t), t)$,

$$\delta L = -\frac{e}{c}\left(\frac{\partial}{\partial t} + \frac{dx^i}{dt}\frac{\partial}{\partial x^i}\right)\Lambda$$

$$= -\frac{e}{c}\frac{d}{dt}\Lambda$$

& the result then follows from part (a).

(a) only

12.3 A particle with mass m and charge e moves in a uniform static electric field $\bar{E}_0$.

a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\bar{v}_0$ was perpendicular to the electric field.

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta \qquad (12.3)$$

For a constant electric field $\bar{E}_0$ with $\bar{B} = 0$, this becomes

$$\frac{dU^0}{d\tau} = \frac{e}{mc} F^{0\beta} U_\beta = \frac{e}{mc} \bar{E}_0 \cdot \bar{u} \quad \text{since } F^{0i} = -E_i,\ U_i = -(\bar{u})_i$$

$$\frac{du^i}{d\tau} = \frac{e}{mc} F^{i0} U_0 \Rightarrow \frac{d\bar{u}}{d\tau} = \frac{e}{mc} \bar{E}_0 U^0$$

where $U^\alpha = (U^0, \bar{u})$

$$\Rightarrow \frac{d^2 U^0}{d\tau^2} = \frac{d}{d\tau}\left(\frac{e}{mc} \bar{E}_0 \cdot \bar{u}\right) = \left(\frac{e}{mc}\right)^2 \vec{E}_0^{\,2}\, U^0$$

$$\Rightarrow U^0 = A \exp\left(\frac{eE_0}{mc}\tau\right) + B \exp\left(-\frac{eE_0}{mc}\tau\right)$$

for constants A, B.

$$\Rightarrow \frac{d\bar{u}}{d\tau} = \frac{e}{mc} \bar{E}_0 U^0 = \frac{e}{mc} \bar{E}_0 \left(A e^{\frac{eE_0}{mc}\tau} + B e^{-\frac{eE_0}{mc}\tau}\right)$$

$$\Rightarrow \bar{u} = \frac{\bar{E}_0}{E_0}\left(A e^{\frac{eE_0}{mc}\tau} - B e^{-\frac{eE_0}{mc}\tau}\right) + \bar{u}_0$$

(cont'd)

12.3 (a), cont'd

Since in integrating we have introduced integration constant, we need to check that

$$\frac{dU^0}{d\tau} = \frac{e}{mc}\bar{E}_0 \cdot \bar{u}$$

$$\Rightarrow \frac{eE_0}{mc}\left(Ae^{\frac{eE_0}{mc}\tau} - Be^{-\frac{eE_0}{mc}\tau}\right) = \frac{e}{mc}\bar{E}_0\cdot\hat{E}_0\left(Ae^{\frac{eE_0}{mc}\tau} - Be^{-\frac{eE_0}{mc}\tau}\right) + \frac{e}{mc}\bar{E}_0\cdot\bar{u}_0$$

$$\Rightarrow \underline{\bar{E}_0\cdot\bar{u}_0 = 0}$$

Let's assume that $\{\tau=0\}$ corresponds to $\{t=0\}$.
We're told that the initial velocity is perpendicular to the electric field

$$\Rightarrow \bar{E}_0\cdot\bar{u}(\tau=0) = 0$$

$$\Rightarrow \bar{E}_0\cdot\hat{E}_0\left(A-B\right) + \underbrace{\bar{E}_0\cdot\bar{u}_0}_{=0} = 0$$

$$\Rightarrow \underline{A=B}$$

hence
$$U^0 = A\left(e^{\frac{eE_0}{mc}\tau} + e^{-\frac{eE_0}{mc}\tau}\right) = 2A\cosh\left(\frac{eE_0}{mc}\tau\right)$$

$$\bar{u} = \hat{E}_0 A\left(e^{\frac{eE_0}{mc}\tau} - e^{-\frac{eE_0}{mc}\tau}\right) + \bar{u}_0$$

$$= 2A\hat{E}_0\sinh\left(\frac{eE_0}{mc}\tau\right) + \bar{u}_0$$

12.3 (a), cont'd

Another requirement: $U_\alpha U^\alpha = c^2$

$$\Rightarrow (U^0)^2 - \bar{u}^2 = c^2$$

$$\Rightarrow A^2\left(\cancel{e^{2\frac{eE_0}{mc}\tau}} + \cancel{e^{-2\frac{eE_0}{mc}\tau}} + 2\right)$$

$$- A^2\left(\cancel{e^{2\frac{eE_0}{mc}\tau}} + \cancel{e^{-2\frac{eE_0}{mc}\tau}} - 2\right) + \overbrace{2\bar{u}_0 \cdot \hat{E}_0}^{=0} A\left(e^{\frac{eE_0}{mc}\tau} - e^{-\frac{eE_0}{mc}\tau}\right)$$

$$+ \bar{u}_0^2$$

$$= 4A^2 + \bar{u}_0^2$$

must $= c^2 \quad \Rightarrow \quad 2A = c(1 + u_0^2/c^2)^{1/2}$

Convert $\bar{u}(\tau=0)$ to an ordinary velocity $\bar{v}_0$:

$$\bar{u}(t=0) = \gamma_0 \bar{v}_0 \quad \text{where} \quad \gamma_0^2 = (1 - v_0^2/c^2)^{-1}$$

Putting this together,

$$\boxed{\begin{aligned} U^0 &= \gamma_0 c \cosh\left(\frac{eE_0}{mc}\tau\right) \\ \bar{u} &= \gamma_0 c \hat{E}_0 \sinh\left(\frac{eE_0}{mc}\tau\right) + \gamma_0 \bar{v}_0 \end{aligned}}$$

Now find position ...

12.3 (a), cont'd

Find position...

$$ct = x^0 = \int_0^\tau U^0 \, d\tau' = \gamma_0 c \left(\frac{mc}{eE_0}\right) \sinh\left(\frac{eE_0}{mc}\tau\right)$$

hence $\tau = \frac{mc}{eE_0} \operatorname{arcsinh}\left(\frac{eE_0 t}{\gamma_0 mc}\right)$

$$U^0 = \gamma_0 c \cosh\left(\operatorname{arcsinh}\left(\frac{eE_0 t}{\gamma_0 mc}\right)\right)$$
$$= \gamma_0 c \left(1 + \left(\frac{eE_0 t}{\gamma_0 mc}\right)^2\right)^{1/2}$$

$$\gamma = \frac{U^0}{c} = \gamma_0 \left(1 + \left(\frac{eE_0 t}{\gamma_0 mc}\right)^2\right)^{1/2}$$

$$\bar{u} = \gamma_0 \cancel{c} \hat{E}_0 \left(\frac{eE_0 t}{\gamma_0 m\cancel{c}}\right) + \gamma_0 \bar{v}_0 = \gamma_0 \left[\frac{e\bar{E}_0 t}{m\gamma_0} + \bar{v}_0\right]$$

$$\boxed{\bar{v}(t) = \frac{\bar{u}}{\gamma} = \left[1 + \left(\frac{eE_0 t}{\gamma_0 mc}\right)^2\right]^{-1/2} \left[\frac{e\bar{E}_0 t}{m\gamma_0} + \bar{v}_0\right]}$$

$$\boxed{\begin{aligned}\bar{x}(t) = \int_0^t \bar{v}(t')\,dt' &= \bar{v}_0 \left(\frac{\gamma_0 mc}{eE_0}\right) \operatorname{arcsinh}\left(\frac{eE_0 t}{\gamma_0 mc}\right) \\ &+ \frac{e\bar{E}_0}{m\gamma_0}\left[\left(\frac{\gamma_0 mc}{eE_0}\right)^2\right]\left[1 + \left(\frac{eE_0 t}{\gamma_0 mc}\right)^2\right]^{+1/2} + \bar{X}_0\end{aligned}}$$