12.14 An alternative Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{8\pi}\, \partial_\alpha A_\beta\, \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$$

a) Derive the Euler-Lagrange equations of motion. Are they Maxwell equations? Under what assumptions?

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{2}{8\pi}\, \partial_\beta A_\alpha$$

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$$

$$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = \frac{\partial \mathcal{L}}{\partial A^\alpha} \Rightarrow -\frac{1}{4\pi}\partial^\beta \partial_\beta A_\alpha = -\frac{1}{c} J_\alpha$$

$$\text{or } \partial^\beta \partial_\beta A_\alpha = \frac{4\pi}{c} J_\alpha$$

We would ordinarily ~~wout~~ write the Maxwell eqn's as

$$\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$$

where $\text{LHS} = \partial^\beta F_{\beta\alpha} = \partial^\beta \partial_\beta A_\alpha - \partial^\beta \partial_\alpha A_\beta$

Note that in Lorenz gauge, $\partial^\beta A_\beta = 0$,

so in Lorenz gauge, the Maxwell eqn's become

$$\partial^\beta \partial_\beta A_\alpha = \frac{4\pi}{c} J_\alpha$$

Hence the Lagrangian above is a valid description in Lorenz gauge.

12.14, cont'd

b) Show explicitly, and with what assumptions, that this Lagrangian density differs from

$\mathcal{L}_0 \quad -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$

by a 4-divergence. Does this added 4-divergence affect the action or the equations of motion?

$$\mathcal{L}_0 = -\frac{1}{16\pi}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) - \frac{1}{c} J_\alpha A^\alpha$$

$$= -\frac{1}{16\pi}\left(2\, \partial_\alpha A_\beta\, \partial^\alpha A^\beta - 2\, \partial_\beta A_\alpha\, \partial^\alpha A^\beta\right) - \frac{1}{c} J_\alpha A^\alpha$$

$$= \underbrace{-\frac{1}{8\pi}\, \partial_\alpha A_\beta\, \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha}_{=\mathcal{L}} + \frac{1}{8\pi}\, \partial_\beta\left(A_\alpha\, \partial^\alpha A^\beta\right) - \frac{1}{8\pi} A_\alpha\, \partial^\alpha \partial_\beta A^\beta$$

In Lorenz gauge, the last term vanishes, & $\mathcal{L}_0$ differs from $\mathcal{L}$ by a total divergence.

The action is unchanged by a total divergence, hence the equations of motion are also unchanged.

(a), (b) only

12.16 a) Starting with the Proca Lagrangian density

$$\mathcal{L}_{Proca} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha$$

and following the same procedure as for electromagnetic fields, show that the symmetric stress-energy-momentum tensor for Proca field is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\gamma} F_{\gamma\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\lambda\nu} F^{\lambda\nu} + \mu^2 \left(A^\alpha A^\beta - \frac{1}{2} g^{\alpha\beta} A_\lambda A^\lambda \right) \right]$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L}$$

$$= \frac{1}{4\pi} F_\lambda{}^\mu \partial^\nu A^\lambda - g^{\mu\nu} (\mathcal{L})$$

Use $\partial^\nu A^\lambda = F^{\nu\lambda} + \partial^\lambda A^\nu$

$$T^{\mu\nu} = + \frac{1}{4\pi} F_\lambda{}^\mu F^{\nu\lambda} + \frac{1}{4\pi} F_\lambda{}^\mu \partial^\lambda A^\nu + \frac{g^{\mu\nu}}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - g^{\mu\nu} \frac{\mu^2}{8\pi} A_\alpha A^\alpha$$

$$= \frac{1}{4\pi}\Big[F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial^\lambda (F_\lambda{}^\mu A^\nu) - (\partial^\lambda F_\lambda{}^\mu) A^\nu - \frac{\mu^2}{2} g^{\mu\nu} A_\alpha A^\alpha \Big]$$

Use the Proca equation of motion

$$\partial^\lambda F_\lambda{}^\mu + \mu^2 A^\mu = 0 \qquad (\text{for } J_\alpha = 0)$$

12.16 a), cont'd

$$T^{\mu\nu} = \frac{1}{4\pi}\left[g^{\mu\gamma}F_{\gamma\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \mu^2 A^\mu A^\nu - \frac{\mu^2}{2}g^{\mu\nu}A_\alpha A^\alpha\right]$$

$$+ \frac{1}{4\pi}\partial^\lambda\left(F_\lambda{}^\mu A^\nu\right)$$

Define $T_D^{\mu\nu} = \frac{1}{4\pi}\partial^\lambda\left(F_\lambda{}^\mu A^\nu\right)$

$$\Theta^{\mu\nu} = T^{\mu\nu} - T_D^{\mu\nu}$$

~~Furth~~ Furthermore, note that

$$\partial_\mu T_D^{\mu\nu} = \frac{1}{4\pi}\partial_\mu\partial^\lambda\left(F_\lambda{}^\mu A^\nu\right)$$

$$= \frac{1}{4\pi}\partial_\mu\left[-\mu^2 A^\mu A^\nu + F_\lambda{}^\mu\partial^\lambda A^\nu\right] \quad \text{using } \partial_\lambda F_\lambda{}^\mu = -\mu^2 A^\mu$$

$$= \frac{1}{4\pi}\left[-\mu^2\partial_\mu\left(A^\mu A^\nu\right) + \mu^2 A_\lambda\partial^\lambda A^\nu + F_\lambda{}^\mu\partial_\mu\partial^\lambda A^\nu\right]$$

$$= \frac{1}{4\pi}\left[\left(\partial_\lambda A^\mu - \partial^\mu A_\lambda\right)\partial_\mu\partial^\lambda A^\nu\right] \quad \text{using } \partial_\mu A^\mu = 0$$

$$= \frac{1}{4\pi}\left[\partial_\mu\left(\partial_\lambda A^\mu\partial^\lambda A^\nu\right) - \partial^\lambda\left(\partial^\mu A_\lambda\partial_\mu A^\nu\right)\right] \quad \text{using } \partial_\mu A^\mu = 0$$

$$= \frac{1}{4\pi}\left[\partial_\mu\left(\partial_\lambda A^\mu\partial^\lambda A^\nu\right) - \partial_\mu\left(\partial_\lambda A^\mu\partial^\lambda A^\nu\right)\right]$$

$$= 0$$

Thus, $\Theta^{\mu\nu}$ must obey the conservation law.

$$\Rightarrow \boxed{\begin{aligned}\Theta^{\mu\nu} &= T^{\mu\nu} - T_D^{\mu\nu} \\ &= \frac{1}{4\pi}\left[g^{\mu\gamma}F_{\gamma\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \mu^2\left(A^\mu A^\nu - \frac{g^{\mu\nu}}{2}A_\alpha A^\alpha\right)\right]\end{aligned}}$$

12.16, cont'd

b) In the presence of an external source, show that

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{J_\lambda F^{\lambda\beta}}{c}$$

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{4\pi}\partial_\alpha\left[g^{\alpha\gamma}F_{\gamma\lambda}F^{\lambda\beta} + \frac{g^{\alpha\beta}}{4}F_{\mu\nu}F^{\mu\nu} + \mu^2\left(A^\alpha A^\beta - \frac{g^{\alpha\beta}}{2}A_\mu A^\mu\right)\right]$$

$$= \frac{1}{4\pi}\Big[g^{\alpha\gamma}(\partial_\alpha F_{\gamma\lambda})F^{\lambda\beta} + g^{\alpha\gamma}F_{\gamma\lambda}(\partial_\alpha F^{\lambda\beta}) + \frac{g^{\alpha\beta}}{2}(\partial_\alpha F_{\mu\nu})F^{\mu\nu}$$
$$+ \mu^2\left(A^\alpha\partial_\alpha A^\beta - g^{\alpha\beta}(\partial_\alpha A_\mu)A^\mu\right)\Big]$$

Use $\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0$

$$\Rightarrow \frac{g^{\alpha\beta}}{2}(\partial_\alpha F_{\mu\nu})F^{\mu\nu} = -\frac{g^{\alpha\beta}}{2}F^{\mu\nu}(\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu})$$
$$= -g^{\alpha\beta}F^{\mu\nu}\partial_\mu F_{\nu\alpha}$$

Also,

$$g^{\alpha\gamma}F_{\gamma\lambda}\partial_\alpha F^{\lambda\beta} = F^{\alpha\lambda}\partial_\alpha F_\lambda{}^\beta = g^{\delta\beta}F^{\alpha\lambda}\partial_\alpha F_{\lambda\delta}$$
$$= g^{\alpha\beta}F^{\mu\nu}\partial_\mu F_{\nu\alpha}$$

$$\Rightarrow g^{\alpha\gamma}F_{\gamma\lambda}(\partial_\alpha F^{\lambda\beta}) + \frac{1}{2}g^{\alpha\beta}(\partial_\alpha F_{\mu\nu})F^{\mu\nu} = 0$$

12.16 b), cont'd

Use $\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha$ (12.92)

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{4\pi}\left[\left(-\mu^2 A_\lambda + \frac{4\pi}{c} J_\lambda\right) F^{\lambda\beta} + \mu^2\left(A^\alpha \partial_\alpha A^\beta - g^{\alpha\beta}(\partial_\alpha A_\mu) A^\mu\right)\right]$$

$$= \frac{1}{c} J_\lambda F^{\lambda\beta} + \frac{\mu^2}{4\pi}\left[-\cancel{A_\lambda \partial^\lambda A^\beta} - \cancel{A_\lambda \partial^\beta A^\lambda} + \cancel{A^\alpha \partial_\alpha A^\beta} - \cancel{g^{\alpha\beta}(\partial_\alpha A_\mu) A^\mu}\right]$$

$$\boxed{= \frac{J_\lambda F^{\lambda\beta}}{c}}$$