3.5 A hollow sphere of inner radius a has the potential specified on its surface to be $\Phi = V(\theta, \phi)$. Prove the equivalence of the following two expressions for the potential:

a) $$\Phi(\bar{x}) = \frac{a(a^2-r^2)}{4\pi} \int \frac{V(\theta', \phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\, d\Omega'$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

b) $$\Phi(\bar{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} \left(\frac{r}{a}\right)^{\ell} Y_{\ell m}(\theta, \phi)$$

where $A_{\ell m} = \int d\Omega'\, Y^*_{\ell m}(\theta', \phi')\, V(\theta', \phi')$

Expression (a) was derived in § 2.6 from the Green f'n

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{a}{x'\left|\bar{x} - \frac{a^2}{x'^2}\bar{x}'\right|}$$

for the potential <u>inside</u> a sphere of radius a.

Expanding in spherical harmonics, from (3.70),

$$\frac{1}{|\bar{x}-\bar{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

In our application, $\bar{x}'$ will lie on the sphere, so $r < r' = a$, so $r_< = r$, $r_> = r'$.

$$\frac{a}{x'\left|\bar{x} - \frac{a^2}{x'^2}\bar{x}'\right|} = \frac{4\pi a}{r'} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

In our application, $r_< = r$, $r_> = \frac{a^2}{r'^2} r' = a^2/r'$

3.5, cont'd

Given that there is no exterior charge distribution, $\rho = 0$, we have

$$\Phi(\bar{x}) = -\frac{1}{4\pi}\int_S \Phi(\bar{x}')\,\frac{\partial G(\bar{x},\bar{x}')}{\partial n'}\,da'$$

Here, for $\bar{x}'$ on the ~~interior~~ boundary sphere S,

$$G(\bar{x},\bar{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r^l}{r'^{\,l+1}} - \frac{a}{r'}\frac{r^l}{(a^2/r')^{l+1}}\right]Y^*_{lm}(\theta',\phi')\cdot Y_{lm}(\theta,\phi)$$

$$= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r^l}{r'^{\,l+1}} - \frac{1}{a}\left(\frac{rr'}{a^2}\right)^l\right]Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$\frac{\partial G}{\partial n'} = +\frac{\partial G}{\partial r'}$$ (since normal points outward)

$$= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[-\frac{(l+1)\,r^l}{r'^{\,l+2}} - \frac{l}{a}\left(\frac{r}{a^2}\right)^l r'^{\,l-1}\right]\cdot Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[-(l+1)\frac{r^l}{a^{l+2}} - l\,\frac{r^l}{a^{l+2}}\right]Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

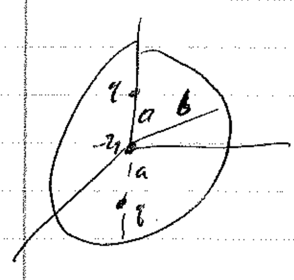
$$= -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{a^{l+2}}\,Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$\Rightarrow \boxed{\Phi(\bar{x}) = -\frac{1}{4\pi}(-4\pi)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int V(\theta',\phi')\,\frac{r^l}{a^{l+\cancel{2}}}\,Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\,a^{\cancel{2}}\,d\Omega'}$$

$$\boxed{= +\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\int V(\theta',\phi')\,Y^*_{lm}(\theta',\phi')\,d\Omega'\right]\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}$$

matching (b)

3.7 Three point charges $(q, -2q, q)$ are located in a straight line with ~~separation~~ separation a and with the middle charge $(-2q)$ at the origin of a grounded conducting spherical shell of radius b.



a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

$$\boxed{\Phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\bar{x} - a\hat{z}|} - \frac{2}{|\bar{x}|} + \frac{1}{|\bar{x} + a\hat{z}|}\right]}$$

For small a,

$$\frac{1}{|\bar{x} - a\hat{z}|} = \frac{1}{|\bar{x}|} - a\hat{z}\cdot\nabla\frac{1}{|\bar{x}|} + \frac{1}{2!}(a\hat{z})_i (a\hat{z})_j \partial_i \partial_j \frac{1}{|\bar{x}|} + \mathcal{O}(a^3)$$

$$= \frac{1}{|\bar{x}|} - a\frac{z}{|\bar{x}|^3} + \frac{a^2}{2!}\left(-\frac{1}{|\bar{x}|^3} + \frac{3z^2}{|\bar{x}|^5}\right) + \mathcal{O}(a^3)$$

$$\frac{1}{|\bar{x} - a\hat{z}|} + \frac{1}{|\bar{x} + a\hat{z}|} = \frac{2}{|\bar{x}|} + a^2\left(-\frac{1}{|\bar{x}|^3} + \frac{3z^2}{|\bar{x}|^5}\right) + \mathcal{O}(a^4)$$

$$\Rightarrow \Phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0}\left[a^2\left(-\frac{1}{|\bar{x}|^3} + \frac{3z^2}{|\bar{x}|^5}\right) + \mathcal{O}(a^4)\right]$$

$$\to \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{|\bar{x}|^3} + \frac{3z^2}{|\bar{x}|^5}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r^3} + \frac{3\cos^2\theta}{r^3}\right] \quad \boxed{= \frac{Q}{2\pi\varepsilon_0}\frac{1}{r^3}P_2(\cos\theta)}$$

3.7, cont'd

b) The presence of the grounded sphere of radius b alters the potential for $r<b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r=b$ or by image charges located at $r>b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r<a$ and $r>a$. Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\phi) \to \frac{Q}{2\pi\varepsilon_0 r^3}\left[1-\frac{r^5}{b^5}\right]P_2(\cos\theta)$$

Recall for a sphere of radius b, from § 2.6,

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{b}{x'\left|\bar{x}-\frac{b^2}{x'^2}\bar{x}'\right|}$$

and the expansion (3.38)

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

Here, the potential due to the three point charges inside the sphere is

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left[q\,G(\bar{x},a\hat{z}) - 2q\,G(\bar{x},0) + q\,G(\bar{x},-a\hat{z})\right]$$

3.7 b), cont'd

Since the point charges all lie along the z axis, $\gamma = \theta$ or $\pi - \theta$, so we expand

$$G(\bar{x}, a\hat{z}) = \frac{1}{|\bar{x} - a\hat{z}|} - \frac{b}{a\left|\bar{x} - \frac{b^2}{a^2} a\hat{z}\right|}$$

$$= \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta) - \frac{b}{a} \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$\hookrightarrow r_> = \max(r, a)$

$\hookrightarrow r_> = \max(r, b^2/a)$

since $b > r$ & $b/a > 1$, $b(b/a) > r$

$$= \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta) - \frac{b}{a} \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{(b^2/a)^{\ell+1}} P_\ell(\cos\theta)$$

$$G(\bar{x}, -a\hat{z}) = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\underbrace{\cos(\pi-\theta)}_{= -\cos\theta}) - \frac{b}{a} \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{(b^2/a)^{\ell+1}} P_\ell(\underbrace{\cos(\pi-\theta)}_{= -\cos\theta})$$

3.7 b), cont'd

Computing $G(\bar{x}, 0)$ can be slightly more subtle, since $x' = 0$.

Let's quickly ~~return to the method of images~~ think this through.

For a point charge at y,

$$G(\bar{x},\bar{y}) = \frac{1}{|\bar{x}-\bar{y}|} - \frac{b/y}{|\bar{x} - \frac{b^2}{y^2}\bar{y}|}$$

$$= \frac{1}{|\bar{x}-\bar{y}|} - \frac{b}{|y\bar{x} - b^2\hat{y}|}$$

so as $y = |\bar{y}| \to 0$,

$$G(\bar{x},\bar{y}) \to \frac{1}{|\bar{x}|} - \frac{b}{|-b^2\hat{y}|} = \frac{1}{|\bar{x}|} - \frac{1}{b}$$

so if our charge is at the origin, then to get $\Phi = 0$ on the sphere, we just shift Φ by a constant.

3.7 b), cont'd

$$\Phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0}\left[G(\bar{x}, a\hat{z}) - 2G(\bar{x}, 0) + G(\bar{x}, -a\hat{z})\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}}\left(1+(-)^{\ell}\right)P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \frac{(ra)^{\ell}}{b^{2\ell+1}}\left(1+(-)^{\ell}\right)P_\ell(\cos\theta) - 2\left(\frac{1}{r} - \frac{1}{b}\right)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\sum_{\ell\ \mathrm{even}} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta) - \sum_{\ell\ \mathrm{even}} \frac{(ra)^{\ell}}{b^{2\ell+1}} P_\ell(\cos\theta) - \frac{1}{r} + \frac{1}{b}\right]$$

$$r_> = \max(a, r)$$

$$r_< = \min(a, r)$$

3.7 b), cont'd

Now, expand for small a:

$$\Phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\Big[\cancel{\frac{1}{r} P_0(\cos\theta)} + \frac{a^2}{r^3} P_2(\cos\theta)$$

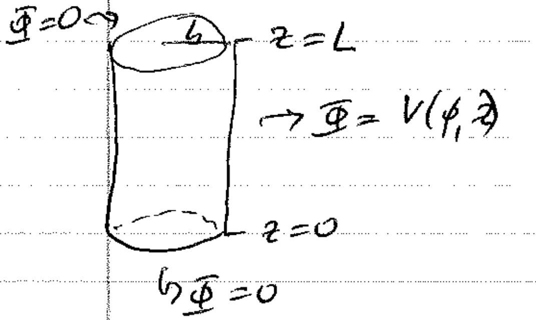
$$- \cancel{\frac{1}{b} P_0(\cos\theta)} - \frac{(ra)^2}{b^5} P_2(\cos\theta)$$

$$- \cancel{\frac{1}{r}} + \cancel{\frac{1}{b}} + O(a^4)\Big]$$

$$= \frac{q a^2}{2\pi\varepsilon_0}\left[\frac{1}{r^3} - \frac{r^2}{b^5}\right] P_2(\cos\theta) + O(a^4)$$

$$= \frac{Q}{2\pi\varepsilon_0}\frac{1}{r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta) + \dots$$

3.9 A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z=0, L$. The potential on the ends is zero, while the potential on the side is $V(\phi, z)$. Using the appropriate separation of variables in cylindrical ~~coor~~ coordinates, find a series solution for the potential anywhere inside the cylinder.



$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\Phi = R(\rho)\, Q(\phi)\, Z(z)$$

$$\Rightarrow \frac{1}{Z}\frac{d^2 Z}{dz^2} = \text{constant.}$$

If $= +k^2$, then sol'ns $\propto e^{\pm kz}$, but cannot vanish at both $z=0$ & $z=L$.
So, instead, take $= -k^2$, so sol'ns $\propto e^{\pm ikz}$

$$\Rightarrow \frac{1}{Q}\frac{d^2 Q}{d\phi^2} = -m^2 \Rightarrow Q(\phi) \propto e^{\pm im\phi}$$

Require $Q(\phi + 2\pi) = Q(\phi) \Rightarrow m$ an integer

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{m^2}{\rho^2}\right) = 0$$

$$\Rightarrow R(\rho) \propto I_m(k\rho),\ K_m(k\rho) \quad \text{(modified Bessel f'ns)}$$

Since we need the sol'ns to be well-behaved at $\rho = 0$, only $I_m(k\rho)$ can contribute.

3.9, cont'd

Also note that $Z(z=0) = Z(z=L) = 0$ forces

$$Z \propto \sin\left(\frac{n\pi z}{L}\right) \text{ for some integer } n, \text{ ie, } k = \frac{n\pi}{L}$$

Thus, the general solution satisfying $\Phi(z=0) = \Phi(z=L) = 0$ for the potential inside the cylinder is of the form

$$\boxed{\Phi(\rho, \phi, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (A_{mn} \sin m\phi + B_{mn} \cos m\phi) \sin\left(\frac{n\pi z}{L}\right) I_m\left(\frac{n\pi}{L}\rho\right)}$$

Next, require

$$V(\phi, z) = \Phi(\rho, \phi, z)\big|_{\rho=b}$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (A_{mn} \sin m\phi + B_{mn} \cos m\phi) \sin\left(\frac{n\pi z}{L}\right) I_m\left(\frac{n\pi b}{L}\right)$$

$$\Rightarrow \boxed{\begin{aligned} A_{mn} &= \frac{1}{\pi} \frac{2}{L} \frac{1}{I_m\left(\frac{n\pi b}{L}\right)} \int_0^{2\pi} d\phi \int_0^L dz \, V(\phi, z) \sin m\phi \sin \frac{n\pi z}{L} \\ B_{mn} &= \frac{1}{\pi} \frac{2}{L} \frac{1}{I_m\left(\frac{n\pi b}{L}\right)} \int_0^{2\pi} d\phi \int_0^L dz \, V(\phi, z) \cos m\phi \sin \frac{n\pi z}{L} \end{aligned}}$$