3.17 The Dirichlet Green function for the unbounded space between the planes at $z=0$ and $z=L$ allows a discussion of a point charge or a distribution of charge between parallel conducting plates held at zero potential.

a) Using cylindrical coordinates show that one form of the Green function is

$$G(\bar{x},\bar{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right)$$

$$\cdot K_m\left(\frac{n\pi}{L}\rho_>\right)$$

We will roughly follow § 3.11.
For a Green function,

$$\nabla^2 G(\bar{x},\bar{x}') = -4\pi\,\delta^3(\bar{x}-\bar{x}')$$

$$= -4\pi\,\frac{\delta(\rho-\rho')}{\rho}\,\delta(\phi-\phi')\,\delta(z-z')$$

where

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \qquad (3.139)$$

and for the boundary conditions here,

$$\delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)$$

where the $\delta(z-z')$ has been expressed in a basis of functions that vanish at $z=0$ & $z=L$.

3.17 a), cont'd

Expand the Green function as

$$G(\bar{x},\bar{x}') = \frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) g_m(n,\rho,\rho')$$

and plug into $\nabla^2 G(\bar{x},\bar{x}') = -4\pi\,\delta^3(\bar{x}-\bar{x}')$:

$$\nabla^2 G = \frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \cdot$$

$$\left.\left[\frac{\partial^2 g_m}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial g_m}{\partial\rho} - \frac{m^2}{\rho^2}g_m - \left(\frac{n\pi}{L}\right)^2 g_m\right]\right\} *$$

Define $k_n = \left(\frac{n\pi}{L}\right)$, $x_n = k_n\rho$,

so

$$* = k_n^2\left[\frac{\partial^2 g_m}{\partial x_n^2} + \frac{1}{x_n}\frac{\partial g_m}{\partial x_n} - \left(1+\frac{m^2}{x^2}\right)g_m\right]$$

& $\nabla^2 G = -4\pi\delta^3$ becomes

$$k_n^2\left(\frac{\partial^2 g_m}{\partial x^2} + \frac{1}{x}\frac{\partial g}{\partial x} - \left(1+\frac{m^2}{x^2}\right)g\right) = -4\pi\frac{\delta(\rho-\rho')}{\rho}$$

For $\rho\neq\rho'$, the sol'ns are modified Bessel f'ns I_m, K_m.
We want the solution to be finite as $\rho\to 0$,
& more generally $G(x,x')$ to be finite everywhere.
However, $K_m(x)$ diverges as $x\to 0$.

Following § 3.11, to have a solution satisfying these conditions & which is also symmetric between $\bar{x}$, $\bar{x}'$, write

$$g_m(n,\rho,\rho') = A\,I_m(k_n\rho_<)\,K_m(k_n\rho_>)$$

3.17 a), cont'd

Plug that ansatz for g_m into the differential equation:

$$\frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} - \left(k^2 + \frac{m^2}{\rho^2}\right) g = -4\pi \frac{\delta(\rho-\rho')}{\rho}$$

Integrate both sides from $\rho' - \varepsilon$ to $\rho' + \varepsilon$ & take a limit as $\varepsilon \to 0$.

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'+\varepsilon} - \left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'-\varepsilon} = -\frac{4\pi}{\rho'} \qquad (*)$$

Now,

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'+\varepsilon} = \frac{\partial}{\partial \rho}\left[A\, I_m(k\rho')\, K_m(k\rho)\right]\Big|_{\rho'+\varepsilon}$$

$$= A\, I_m(k\rho')\,(k)\, K_m'(k\rho')$$

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'-\varepsilon} = \frac{\partial}{\partial \rho}\left[A\, I_m(k\rho)\, K_m(k\rho')\right]\Big|_{\rho'-\varepsilon}$$

$$= A(k)\, I_m'(k\rho)\, K_m(k\rho')$$

Hence

$$(*) \Rightarrow Ak\left(I_m K_m' - I_m' K_m\right) = -\frac{4\pi}{\rho'}$$

We can evaluate this at any convenient ρ' to solve for A.

3.17 a), cont'd

S'pose $k\rho' \gg 1$.

$$I_m(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \quad K_m(x) \simeq \sqrt{\frac{\pi}{2x}}\, e^{-x} \quad \text{for } x \gg 1$$

In this regime,

$$I_m'(x) \simeq \frac{e^x}{\sqrt{2\pi x}} - \frac{1}{2}\frac{e^x}{\sqrt{2\pi}} x^{-3/2}$$

$$K_m'(x) \simeq -\sqrt{\frac{\pi}{2x}}\, e^{-x} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\, x^{-3/2} e^{-x}$$

$$I_m K_m' - I_m' K_m \simeq \frac{e^x}{\sqrt{2\pi x}}\left(-\sqrt{\frac{\pi}{2x}}\, e^{-x} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\, x^{-3/2} e^{-x}\right)$$

$$- \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(\frac{e^x}{\sqrt{2\pi x}} - \frac{1}{2}\frac{1}{\sqrt{2\pi}} x^{-3/2}\right)$$

$$= -\frac{1}{2x} - \cancel{\frac{1}{2}\frac{1}{2x^2}} - \frac{1}{2x} + \cancel{\frac{1}{2}\frac{1}{2x^2}}$$

$$= -\frac{1}{x}$$

Thus,

$$Ak\left(I_m K_m' - I_m' K_m\right) = -\frac{1}{k\rho'}(Ak) = -\frac{A}{\rho'}$$

$$\text{Demand} = -\frac{4\pi}{\rho'}$$

$$\Rightarrow A = 4\pi$$

3.17 a), cont'd

Putting this together, we see

$$\boxed{G(\bar{x},\bar{x}') = \frac{4\cancel{\pi}}{\cancel{\pi}L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m(k_n\rho_<)K_m(k_n\rho_>) \quad \text{for } k_n = \frac{n\pi}{L}}$$

as expected.

3.17, cont'd

b) Show that an alternative form of the Green function is

$$G(\bar{x},\bar{x}') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dk\; e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}$$

For a Green function,

$$\nabla_x^2 G(\bar{x},\bar{x}') = -4\pi\,\delta^3(\bar{x}-\bar{x}')$$

$$= -4\pi\,\frac{\delta(\rho-\rho')}{\rho}\,\delta(\phi-\phi')\,\delta(z-z')$$

where $\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$ (3.139)

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty dk\; k\, J_m(k\rho) J_m(k\rho') \quad (3.108)$$

Expand the Green function as

$$G(\bar{x},\bar{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\; e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho')\, g_m(k,z,z')$$

and plug into $\nabla^2 G = -4\pi\delta^3$:

$$\nabla_x^2 G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho') \quad 2$$

$$\hookrightarrow \left[k^2 J_m''(k\rho) g_m + \frac{k}{\rho} J_m'(k\rho) g_m - \frac{m^2}{\rho^2} J_m(k\rho)\, g_m + J_m \frac{\partial^2 g_m}{\partial z^2}\right]$$

3.17(b), cont'd

$$\nabla_x^2 G(x, x') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho')\, 2$$

$$G\left[k^2\underbrace{\left(J_m'' + \frac{1}{k\rho}J_m' - \frac{m^2}{k^2\rho^2}J_m\right)}_{-J_m} g_m + J_m\frac{\partial^2 g_m}{\partial z^2}\right]$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho')\, J_m(k\rho)\left[\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m\right]$$

Demand

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\, k\, J_m(k\rho')J_m(k\rho)\, \delta(z-z')\, (-4\pi)$$

Hence

$$\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m = -4\pi k\, \delta(z-z')$$

For $z \neq z'$, sol'ns $\propto e^{\pm kz}$

Require $g_m = 0$ for $z=0$ & $z=L$, & symmetry between $\bar{x}, \bar{x}'$.

Vanishing suggests if $z < z'$, $g_m \propto \sinh kz$

if $z > z'$, $g_m \propto \sinh k(L-z)$

so from symmetry, write

$$g_m(k, z, z') = A \sinh(kz_<) \sinh k(L - z_>)$$

for some constant A.

3.(17 b), cont'd

We can determine the constant A by solving the differential equation

$$\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m = -4\pi k\, \delta(z-z')$$

Integrating from $z = z'-\varepsilon$ to $z = z'+\varepsilon$ & taking a limit as $\varepsilon \to 0$, we get

$$\left.\frac{\partial g_m}{\partial z}\right|_{z=z'+\varepsilon} - \left.\frac{\partial g_m}{\partial z}\right|_{z=z'-\varepsilon} = -4\pi k$$

Now,

$$\left.\frac{\partial g_m}{\partial z}\right|_{z=z'+\varepsilon} = \left.\frac{\partial}{\partial z}\Big(A \sinh(kz') \sinh k(L-z) \Big)\right|_{z=z'+\varepsilon}$$

$$= -Ak \sinh(kz') \cosh k(L-z')$$

$$\left.\frac{\partial g_m}{\partial z}\right|_{z=z'-\varepsilon} = \left.\frac{\partial}{\partial z}\Big(A \sinh(kz) \sinh k(L-z') \Big)\right|_{z=z'-\varepsilon}$$

$$= +Ak \cosh(kz') \sinh k(L-z')$$

$$\left.\frac{\partial g_m}{\partial z}\right|_{z'+\varepsilon} - \left.\frac{\partial g_m}{\partial z}\right|_{z'-\varepsilon} = -Ak\Big(\sinh kz' \cosh k(L-z') + \cosh kz' \sinh k(L-z') \Big)$$

$$= -Ak \sinh\big(kz' + k(L-z') \big)$$

$$= -Ak \sinh(kL)$$

Demand $= -4\pi k$

$$\Rightarrow A = +\frac{4\pi}{\sinh(kL)}$$

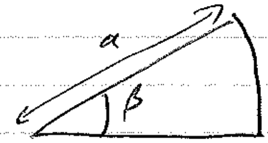
3.17 b), cont'd

Putting this together, we see

$$G(\bar{x}, \bar{x}') = \frac{4\pi}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<) \sinh k(L-z_>)}{\sinh kL}$$

$$= 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<) \sinh k(L-z_>)}{\sinh kL}$$

3.22 The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi=0$, $\phi=\beta$, $\rho=a$, as indicated below:



Using separation of variables in polar coordinates, show that the Green function can be written as

$$G(\rho,\phi,\rho',\phi') = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right) \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

For a Green function, $\nabla_x^2 G(\bar{x},\bar{x}') = -4\pi\, \delta^2(\bar{x}-\bar{x}')$

$$= -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi')$$

where $\delta(\phi-\phi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$

(expanded in a basis of f's that vanish at $\phi=0$ & $\phi=\beta$)

With this in mind, let's make the ansatz

$$G(\bar{x},\bar{x}') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) g_m(\rho,\rho')$$

Plug in:

$$\nabla^2 G = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) \left(\frac{1}{\rho}\frac{\partial g_m}{\partial\rho} + \frac{\partial^2 g_m}{\partial\rho^2} - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g_m \right)$$

& demand $= -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi')$

3.22, cont'd

For $\rho \neq \rho'$,

$$\frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g = 0$$

$$\Rightarrow g \propto \rho^{\pm m\pi/\beta}$$

Boundary conditions:

$$g(\rho = 0) = 0 = g(\rho = a)$$

Thus, for $\rho < \rho'$, $g \propto \rho^{+m\pi/\beta}$ so that $\to 0$ as $\rho \to 0$

& for $\rho > \rho'$, $g \propto \rho^{-m\pi/\beta} - \dfrac{\rho^{+m\pi/\beta}}{a^{2m\pi/\beta}}$, so that $\to 0$ as $\rho \to a$

Require symmetric between $\bar{x}$, $\bar{x}'$:

$$g_m(\rho, \rho') = A \rho_<^{+m\pi/\beta}\left(\rho_>^{-m\pi/\beta} - \frac{\rho_>^{+m\pi/\beta}}{a^{2m\pi/\beta}}\right)$$

To solve for A we return to

$$\frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g = -4\pi \frac{\delta(\rho - \rho')}{\rho}$$

Integrating both sides from $\rho = \rho' - \varepsilon$ to $\rho = \rho' + \varepsilon$ & taking limit $\varepsilon \to 0$, we find

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho' + \varepsilon} - \left.\frac{\partial g}{\partial \rho}\right|_{\rho' - \varepsilon} = -\frac{4\pi}{\rho'}$$

3.22, cont'd

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'+\varepsilon} = \frac{\partial}{\partial \rho}\left(A\rho'^{\,m\pi/\beta}\left(\rho^{-m\pi/\beta} - \frac{\rho^{+m\pi/\beta}}{a^{2m\pi/\beta}}\right)\right)\Bigg|_{\rho=\rho'+\varepsilon}$$

$$= A\rho'^{\,m\pi/\beta}\left(-\frac{m\pi}{\beta}\rho'^{\,-m\pi/\beta-1} - \frac{m\pi}{\beta}\frac{\rho'^{\,m\pi/\beta-1}}{a^{2m\pi/\beta}}\right)$$

$$= -\frac{m\pi}{\beta}A\left(\frac{1}{\rho'} + \frac{1}{\rho'}\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right)$$

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho=\rho'-\varepsilon} = \frac{\partial}{\partial \rho}\left(A\rho^{m\pi/\beta}\left(\rho'^{\,-m\pi/\beta} - \frac{\rho'^{\,+m\pi/\beta}}{a^{2m\pi/\beta}}\right)\right)\Bigg|_{\rho'-\varepsilon}$$

$$= +\frac{m\pi}{\beta}A\left(\frac{1}{\rho'} - \frac{1}{\rho'}\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right)$$

$$\left.\frac{\partial g}{\partial \rho}\right|_{\rho'+\varepsilon} - \left.\frac{\partial g}{\partial \rho}\right|_{\rho'-\varepsilon} = -\frac{2m\pi}{\beta}A\frac{1}{\rho'}$$

$$\text{Demand} = -\frac{4\pi}{\rho'}$$

$$\Rightarrow \underline{A = \frac{2\beta}{m}}$$

$$\Rightarrow \boxed{G(\bar{x},\bar{x}')} = \sum_{m=1}^{\infty}\frac{2}{\beta}\frac{2\beta}{m}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)\rho_<^{m\pi/\beta}\left(\rho_>^{-m\pi/\beta} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)$$

$$\boxed{= \sum_{m=1}^{\infty}\frac{4}{m}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)\rho_<^{m\pi/\beta}\left(\rho_>^{-m\pi/\beta} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)}$$

4.2 A point dipole with dipole moment $\bar{p}$ is located at the point $\bar{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{eff}(\bar{x}) = -\bar{p}\cdot\nabla\delta^3(\bar{x}-\bar{x}_0).$$

Let's consider the potential first.
Potential due to a dipole $\bar{p}$ at $\bar{x}_0$ is

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\bar{p}\cdot(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3} \qquad (4.10)$$

Claim $= \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3x'\,\frac{\rho_{eff}(\bar{x}')}{|\bar{x}-\bar{x}'|}$.

Note

$$\frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho_{eff}(\bar{x}')}{|\bar{x}-\bar{x}'|} = -\frac{1}{4\pi\varepsilon_0}\int d^3\bar{x}'\,\frac{\bar{p}\cdot\nabla\delta^3(\bar{x}'-\bar{x}_0)}{|\bar{x}-\bar{x}'|}$$

$$= +\frac{1}{4\pi\varepsilon_0}\int d^3\bar{x}'\,\bar{p}\cdot\nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)\delta^3(\bar{x}'-\bar{x}_0)$$

$$= +\frac{1}{4\pi\varepsilon_0}\int d^3x'\,\bar{p}\cdot\frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}\,\delta^3(\bar{x}'-\bar{x}_0)$$

$$= +\frac{1}{4\pi\varepsilon_0}\,\bar{p}\cdot\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3}$$

so indeed this ρ_{eff} generates the correct potential Φ.

4.2, cont'd

Next, compare the energy of a dipole in an external field.

$$= -\bar{p}\cdot\bar{E}(\bar{x}_0) \qquad (4.24)$$

$$\text{Claim} = \int d^3x \; \rho_{eff}(\bar{x})\,\Phi(\bar{x}) \qquad (4.21)$$

Note

$$\int d^3x \; \rho_{eff}(\bar{x})\,\Phi(\bar{x}) = \int d^3x \,(-)\, \bar{p}\cdot\left(\nabla\delta^3(\bar{x}-\bar{x}_0)\right)\Phi(\bar{x})$$

$$= + \int d^3\bar{x}\; \delta^3(\bar{x}-\bar{x}_0)\; \bar{p}\cdot\nabla\Phi(\bar{x})$$

$$= -\bar{p}\cdot\bar{E}(\bar{x}_0)$$

so the energy of a dipole in an external field is correctly duplicated by this ρ_{eff}.

(a, b) only

4.7 A localized distribution of charge has a charge density

$$\rho(\bar{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moment. Write down the potential at large distances as a finite expansion in Legendre polynomials.

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \qquad (4.1)$$

for large r,
where the multipole moment q_{lm} are given as

$$q_{lm} = \int d^3x' \, Y_{lm}^*(\theta',\phi') \, r'^{l} \rho(\bar{x}') \qquad (4.3)$$

Here,

$$q_{lm} = \frac{1}{64\pi} \int d^3x' \, Y_{lm}^*(\theta',\phi') \, r'^{l+2} e^{-r'} \sin^2\theta'$$

$$= \frac{1}{64\pi} \underbrace{\int_0^\infty dr' \, r'^{l+4} e^{-r'}}_{=\Gamma(l+5) = (l+4)!} \int d\Omega' \, Y_{lm}^*(\theta',\phi') \sin^2\theta'$$

$$Y_{20} = \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2}(2-3\sin^2\theta), \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \sin^2\theta = \frac{2}{3}\sqrt{4\pi}\, Y_{00} - \frac{2}{3}\left(\frac{4\pi}{5}\right)^{1/2} Y_{20}$$

$$\Rightarrow \int d\Omega' \, Y_{lm}^*(\theta',\phi') \sin^2\theta' = \frac{2}{3}(4\pi)^{1/2}\left(\delta_{l0}\delta_{m0} - 5^{-1/2}\delta_{l2}\delta_{m0}\right)$$

(cont'd)

4.7 a), cont'd

$$\boxed{q_{\ell m} = \frac{1}{64\pi}(\ell+4)!\ \frac{2}{3}(4\pi)^{1/2}\left(\delta_{\ell 0}\delta_{m0} - 5^{-1/2}\delta_{\ell 2}\delta_{m0}\right)}$$

Then,

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\, q_{\ell m}\,\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[4\pi\, q_{00}\,\frac{Y_{00}}{r} + \frac{4\pi}{5}\, q_{20}\,\frac{Y_{20}}{r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{64\pi}\frac{2}{3}(4\pi)^{1/2}\left[4\pi(4!)\frac{Y_{00}}{r} - \frac{4\pi}{5}5^{-1/2}(6!)\frac{Y_{20}}{r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{64\pi}\frac{2}{3}(\cancel{4\pi})^{1/2}\left[\frac{4\pi(4!)}{(\cancel{4\pi})^{1/2}}\frac{1}{r} - \frac{4\pi}{5}\cancel{5^{-1/2}}(6!)\left(\frac{\cancel{5}}{\cancel{4\pi}}\right)^{1/2}\frac{\frac{3}{2}\cos^2\theta - \frac{1}{2}}{r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{64\pi}\frac{2}{3}\left[(4!)\frac{P_0(\cos\theta)}{r} - \frac{6!}{5}\frac{P_2(\cos\theta)}{r^3}\right]$$

using $P_0(x) = 1,\quad P_2(x) = \frac{1}{2}(3x^2-1)$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{\cancel{16}}\frac{\cancel{2}}{\cancel{3}}(\cancel{4!})\left[\frac{P_0(\cos\theta)}{r} - 6\,\frac{P_2(\cos\theta)}{r^3}\right]$$

for large r.

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{6\,P_2(\cos\theta)}{r^3}\right]$$

4.7, cont'd

b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]$$

The results of (a) were only valid for large r.

~~Here, we~~

Here, use

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}$$

with $\displaystyle \frac{1}{|\bar{x}-\bar{x}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)$

Thus,

$$(4\pi\varepsilon_0)\Phi(\bar{x}) = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\int d^3x' \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)\frac{1}{64\pi} r'^2 e^{-r'}\sin^2\theta'$$

$$= \frac{4\pi}{64\pi}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1} Y_{\ell 0}(\theta,\phi)\int_0^{\infty} r'^2 dr' \int d\Omega' \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell 0}(\theta',\phi')\, r'^2 e^{-r'}\sin^2\theta'$$

Note $\displaystyle Y_{\ell 0} = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} P_\ell(\cos\theta)$

$$= \frac{4\pi}{64\pi}(2\pi)\sum_{\ell=0}^{\infty}\frac{1}{4\pi}P_\ell(\cos\theta)\int_0^{\infty} dr'\, r'^4 e^{-r'} \frac{r_<^{\ell}}{r_>^{\ell+1}}\int_{-1}^{1} d(\cos\theta')\, P_\ell(\cos\theta')\sin^2\theta'$$

4.7 b), cont'd

$$\int_{-1}^{1} d(\cos\theta')\, P_\ell(\cos\theta')\, \sin^2\theta'$$

$$= \int_{-1}^{1} dx\, P_\ell(x)\,(1-x^2)$$

Now, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_0(x) = 1$

$$\Rightarrow \quad 2P_2(x) + P_0(x) = 3x^2$$

$$= \int_{-1}^{1} dx\, P_\ell(x)\left[P_0(x) - \left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right)\right]$$

$$= \int_{-1}^{1} dx\, P_\ell(x)\left[\frac{2}{3}P_0(x) - \frac{2}{3}P_2(x)\right]$$

Use $\int_{-1}^{1} P_{\ell'}(x)\, P_\ell(x)\, dx = \frac{2}{2\ell+1}\,\delta_{\ell'\ell}$ (3.21)

$$= \frac{2}{3}\left(2\,\delta_{\ell 0} - \frac{2}{5}\,\delta_{\ell 2}\right)$$

4.7 b), cont'd

$$\int_0^\infty dr' \, r'^4 e^{-r'} \frac{r_<^\ell}{r_>^{\ell+1}}$$

$$= \int_0^r dr' \, r'^4 e^{-r'} \frac{r'^\ell}{r^{\ell+1}} + \int_r^\infty dr' \, r'^4 e^{-r'} \frac{r^\ell}{r'^{\ell+1}} = (*)$$

Note that we only need the result for $\ell = 0, 2$.

$\ell = 0$:

$$* = \int_0^r dr' \, \frac{r'^4 e^{-r'}}{r} + \int_r^\infty dr' \, r'^3 e^{-r'}$$

$$= -\frac{1}{r} e^{-r'} \left[r'^4 + 4r'^3 + 4(3)r'^2 + 4(3)(2)r' + 4! \right]_0^r$$

$$- e^{-r'} \left[r'^3 + 3r'^2 + (3)(2)r' + (3!) \right]_r^\infty$$

$$= -\frac{1}{r} e^{-r} \left[\cancel{r^4} + 4r^3 + (4)(3)r^2 + (4)(3)(2)r + 4! \right]$$

$$+ \frac{1}{r}\left[4! \right] + e^{-r} \left[\cancel{r^3} + 3r^2 + (3)(2)r + (3!) \right]$$

$$= \frac{4!}{r} + e^{-r} \left[-r^2 - 6r - 18 - \frac{4!}{r} \right]$$

4.7 b), cont'd

$\ell = 2$:

$$* = \int_0^r dr' \, \frac{r'^6 e^{-r'}}{r^3} + \int_r^\infty dr' \, r' e^{-r'} r^2$$

$$= \frac{-1}{r^3} e^{-r'} \Big[r'^6 + 6r'^5 + (6)(5)r'^4 + (6)(5)(4)r'^3 + (6)(5)(4)(3)r'^2$$

$$+ (6)(5)(4)(3)(2)r' + 6! \Big] \Big|_0^r$$

$$- r^2 e^{-r'} \Big[r' + 1 \Big] \Big|_r^\infty$$

$$= -\frac{1}{r^3} e^{-r} \Big[\cancel{r^6} + 6r^5 + (6)(5)r^4 + (6)(5)(4)r^3 + \frac{6!}{2!} r^2 + 6! \, r + 6! \Big]$$

$$+ \frac{1}{r^3} \Big[6! \Big]$$

$$+ r^2 e^{-r} \Big[\cancel{r} + 1 \Big]$$

$$= \frac{6!}{r^3} - e^{-r} \Big[5r^2 + (6)(5)r + (6)(5)(4) + \frac{6!}{2!} \frac{1}{r} + \frac{6!}{r^2} + \frac{6!}{r^3} \Big]$$

4.7 b), cont'd

Putting this together, we see

$$(4\pi\varepsilon_0)\,\Phi(\bar{x}) = \underbrace{\frac{2\pi}{64\pi}\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\int_0^\infty dr'\, r'^4 e^{-r'}\frac{r_<^\ell}{r_>^{\ell+1}}\int_{-1}^{1} d(\cos\theta')\, P_\ell(\cos\theta')\sin^2\theta'}$$

$$= \frac{1}{32}\,\frac{4}{3}\Bigg[(1)\left(\frac{4!}{r} + e^{-r}\left(-r^2 - 6r - 18 - \frac{4!}{r}\right)\right)$$

$$- \frac{1}{5}P_2(\cos\theta)\left(\frac{6!}{r^3} - e^{-r}\left(5r^2 + (6)(5)r + (6)(5)(4) + \frac{6!}{2}\frac{1}{r} + \frac{6!}{r^2} + \frac{6!}{r^3}\right)\right)\Bigg]$$

$$\boxed{\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{4!}\left[\frac{1}{r}\left(4! - e^{-r}\left(4! + 18r + 6r^2 + r^3\right)\right) - \frac{6P_2(\cos\theta)}{r^3}\left(4! - e^{-r}\left(4! + 4!\,r + 12r^2 + 4r^3 + r^4 + \frac{r^5}{6}\right)\right)\right]}$$

Note in the limit of large r,

$$\Phi(\bar{x}) \simeq \frac{1}{4\pi\varepsilon_0}\,\frac{1}{4!}\left[\frac{4!}{r} - \frac{6P_2(\cos\theta)}{r^3}(4!)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{6P_2(\cos\theta)}{r^3}\right],$$

duplicating the result of (a).

(cont'd)

47 b), cont'd

Next, let's expand $\Phi(\vec{x})$ for small r.

$$\Phi(\vec{x}) \simeq \frac{1}{4\pi\varepsilon_0}\frac{1}{4!}\Bigg[\frac{1}{r}\left(4! - \left(1 - r + \frac{r^2}{2!} - \frac{r^3}{3!} + \frac{r^4}{4!} + \cdots\right)\left(4! + 18r + 6r^2 + r^3\right)\right)$$

$$- \frac{6P_2(\cos\theta)}{r^3}\left(4! - \left(1 - r + \frac{r^2}{2!} - \frac{r^3}{3!} + \frac{r^4}{4!} - \frac{r^5}{5!} + \cdots\right)\left(4! + 4!r + 12r^2 + 4r^3 + r^4 + \frac{r^5}{6}\right)\right)\Bigg]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{4!}\Bigg[\frac{1}{r}\left(\cancel{24} - \left(\cancel{24} - 6r + O(r^5)\right)\right)$$

$$- \frac{6P_2(\cos\theta)}{r^3}\left(\cancel{24} - \left(\cancel{24} - \frac{r^5}{30} + O(r^7)\right)\right)\Bigg]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{24}\Bigg[6 + O(r^4)$$

$$- 6P_2(\cos\theta)\left(\frac{r^2}{30} + O(r^4)\right)\Bigg]$$

$$\boxed{= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta) + O(r^4)\right]}$$

for small r