Physics 5405: Classical electromagnetism I

Fall 2023

Test 1

September 27, 2023

NAME: ______________________________

Solutions

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has 10 pages, including this cover sheet. If it does not, see me.

~~2~~ 1. (15 points) A conducting spherical shell of radius a is held at potential $+V$. The volume outside the shell is filled with electric charge of volume density

$$\rho(\vec{x}) = \rho_0 \exp(-r/a) \sin^2\theta$$

Write down an integral expression for the potential everywhere ~~inside~~ outside the shell. Do not try to compute the integral.

Dirichlet boundary conditions:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}\, da' \qquad (1.44)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_a^{\infty} r'^2\, dr' \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, \rho_0\, e^{-r'/a} \sin^2\theta' \cdot$$

$$\left[\left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{-1/2} - \left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{-1/2} \right]$$

$$- \frac{V}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\,(-)\,\frac{r^2 - a^2}{a\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}}$$

using (2.17), (2.18)

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~~3.~~ ~~(2?~~ points) A line charge with linear charge density τ is placed parallel to and a distance R away from the axis of a conducting cylinder of radius b held at a fixed voltage such that the potential vanishes at infinity.

(a) Find the magnitude and position of the image charge(s).

Potential due to a single line charge: $\Phi = -\frac{\tau}{2\pi\varepsilon_0} \ln |\bar{x} - \bar{x}'|$

Replace the cylinder by an image line charge, linear charge density τ', located along the line connecting the center of the cylinder & τ.

$$\Phi(\bar{x}) = -\frac{\tau}{2\pi\varepsilon_0} \ln|\bar{x} - \bar{R}| - \frac{\tau'}{2\pi\varepsilon_0} \ln|\bar{x} - \bar{R}'|$$

Far from the cylinder, $\Phi(\bar{x}) \simeq -\frac{(\tau + \tau')}{2\pi\varepsilon_0} \ln|\bar{x}|$,

so in order for $\Phi \to 0$, must require $\boxed{\tau' = -\tau}$

Solve for $\bar{R}'$ that makes Φ constant along cylinder:

$$\Phi(\bar{x}) = -\frac{\tau}{2\pi\varepsilon_0} \ln \frac{|\bar{x} - \bar{R}|}{|\bar{x} - \bar{R}'|} \quad \text{constant for } |\bar{x}| = b:$$

$\hat{r}$ = unit vector to τ (not necessarily radial)

$$|\bar{b} - \bar{R}| = |b\hat{b} - R\hat{r}| = R\left|\hat{r} - \frac{b}{R}\hat{b}\right|$$

$$|\bar{b} - \bar{R}'| = |b\hat{b} - R'\hat{r}| = b\left|\hat{b} - \frac{R'}{b}\hat{r}\right| = b\left|\hat{r} - \frac{R'}{b}\hat{b}\right|$$

so if we require $\boxed{\frac{b}{R} = \frac{R'}{b} \text{ or } R' = b^2/R,}$

then $\Phi(\bar{b}) = -\frac{\tau}{2\pi\varepsilon_0} \ln(R/b)$ = constant along cylinder.

(b) (~~??~~ 10 points) Find the force per unit length on the cylinder.

Replace the cylinder with the image charge.

$\boxed{E}$ due to τ $= \frac{\tau}{2\pi\epsilon_0}\frac{1}{r}$ for r = distance from τ

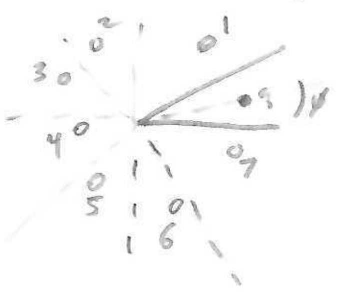
Here,

$$|F| = -\frac{\tau^2}{2\pi\epsilon_0}\left(R - s^2/R\right)^{-1}$$

& directed along the axis connecting center of cylinder to τ.

3. In two dimensions, consider two conducting planes intersecting at an angle $\beta = \pi/4$ (45 degrees). For simplicity, assume one plane is located along the x axis.

   (a) (15 points) A charge q is placed between the planes. What are the locations and magnitudes of all the image charges? (For simplicity, give their positions in polar coordinates.)



S'pose q at (ρ, ϕ)

Image charges:

$q_1 = -q$ at $(\rho, \frac{\pi}{2} - \phi)$

$q_2 = +q$ at $(\rho, \frac{\pi}{2} + \phi)$

$q_3 = -q$ at $(\rho, \pi - \phi)$

$q_4 = +q$ at $(\rho, \pi + \phi)$

$q_5 = -q$ at $(\rho, \frac{3\pi}{2} - \phi)$

$q_6 = +q$ at $(\rho, \frac{3\pi}{2} + \phi)$

$q_7 = -q$ at $(\rho, -\phi = 2\pi - \phi)$

(b) (10 points) What is the Green function for the region between the conducting planes? Use the Green function given in exercise 2.17(a).

Notation: $\bar{x}_0$ = position of original charge at ρ, ϕ

$\bar{x}_i$ = position of i^{th} image charge

For one charge at $\bar{x}'$, from (2.17)(a),

$$G(\bar{x},\bar{x}') = -\ln(\rho^2+\rho'^2 - 2\rho\rho'\cos(\phi-\phi'))$$

Here:

$$G(x,x') = -\sum_{i=0}^{7} (-)^i \ln\left[2\rho^2 - 2\rho^2\cos(\phi-\phi_i)\right]$$

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~~3~~ (~~30~~ points) In two dimensions, one conductor lies along the negative y axis ($x = 0$, $y \leq 0$) and is held at potential 0, and a second conductor lies along the downward-facing parabola

$$y = \frac{1}{2}\left(a^2 - \frac{x^2}{a^2}\right), \tag{1}$$

and is held at constant potential V_0.

Find the most general solution for the scalar potential Φ between the two conductors. Assume there are no charge or current sources present.

It will be convenient to use two-dimensional parabolic coordinates (σ, τ), which are defined by

$$x = \sigma\tau, \tag{2}$$

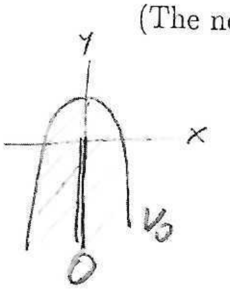
$$y = \frac{1}{2}\left(\tau^2 - \sigma^2\right) = \frac{1}{2}\left(\tau^2 - \frac{x^2}{\tau^2}\right) \tag{3}$$

in terms of rectangular coordinates x, y. (The negative y axis is $\tau = 0$ in these coordinates.) The Laplacian in parabolic coordinates is

$$\nabla^2\Phi = \frac{1}{\sigma^2 + \tau^2}\left(\frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\partial^2\Phi}{\partial\tau^2}\right). \tag{4}$$

Leave your result in terms of parabolic coordinates.

(The next page is left blank as extra space for your solution.)



Separate variables:

$$\Phi(\sigma,\tau) = S(\sigma)\,T(\tau)$$

$$\nabla^2\Phi = 0 \Rightarrow \frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\partial^2\Phi}{\partial\tau^2} = 0$$

$$\Rightarrow \frac{1}{S}S'' + \frac{1}{T}T'' = 0$$

$$\Rightarrow \frac{1}{S}S'' = +m^2, \quad \frac{1}{T}T'' = -m^2 \quad \text{for constant } m$$

Boundary conditions:

$$\Phi(\tau=0) = 0, \qquad \Phi(\tau=a) = V_0$$

We'll first find sol'ns for $m=0$, then consider $m \neq 0$ separately.

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$m=0$

$S = S_0 + S_1\sigma, \quad T = T_0 + T_1\tau \qquad$ for $S_{0,1}$ $T_{0,1}$ constant

We can use this sol'n to solve the (constant) b.c.:

$\Phi(\tau=0)=0 \Rightarrow (S_0+S_1\sigma)(T_0)=0 \Rightarrow T_0=0$

$\Phi(\tau=a)=V_0 \Rightarrow (S_0+S_1\sigma)(T_1 a)=V_0 \Rightarrow S_1=0, \quad S_0 T_1 = \frac{V_0}{a}$

so one sol'n is $\underline{\Phi(\sigma,\tau) = \frac{V_0}{a}\tau}$

Subtracting this sol'n, we can require that $m \neq 0$ modes obey $\Phi(\tau=0)=0=\Phi(\tau=a)$.

Write $T_m(\tau) = A_m \sin m\tau + B_m \cos m\tau$

$\Phi(\tau=0) \propto B_m$ so require $B_m = 0$

$\Phi(\tau=a) \propto A_m \sin ma + 0$

so require $\sin ma = 0 \Rightarrow ma = n\pi$ for integer n

write $S_m(\tau) = C_m e^{+m\sigma} + D_m e^{-m\sigma}$

& we have the general sol'n, a linear combination:

$$\boxed{\Phi(\sigma,\tau) = \frac{V_0}{a}\tau + \sum_{n=1}^{\infty}\left(C_m e^{+\frac{n\pi\sigma}{a}} + D_m e^{-\frac{n\pi\sigma}{a}}\right)\sin\frac{n\pi\tau}{a}}$$