Physics 5405: Classical electromagnetism I

Fall 2023

Test 2

November 6, 2023

NAME: ______________________ Solutions

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has 9 pages, including this cover sheet. If it does not, see me.

1. (a) (~~5~~ 10 points) First, write the most general expression for an azimuthally symmetric potential $\Phi(r, \theta)$ outside a sphere of radius a, i.e. for $r > a$ and independent of ϕ.

Most general sol'n of Laplace's eq'n with azimuthal symmetry:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta) \qquad \text{(Jackson (3.73))}$$

However, we want this to be finite as $r \to \infty$,
hence $A_\ell = 0$ for $\ell > 0$

(b) (~~2?~~ 30 points) A sphere of radius a has azimuthally-symmetric potential

$$V(\theta) = \begin{cases} 0 & \phi < \beta, \\ V_0 & \beta < \phi < \pi - \beta, \\ 0 & \phi > \pi - \beta. \end{cases} \tag{1}$$

Compute the potential outside the sphere (i.e. $r > a$), by applying the boundary condition $\Phi(a, \theta) = V(\theta)$ to the previous expression for the azimuthally-symmetric potential outside a sphere. For full credit, the answer must be expressed without any integrals, meaning all integrals must be evaluated. To that end, use the identity (Jackson (3.28))

$$(2\ell + 1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x), \tag{2}$$

in conventions in which $P_{-1}(x) = 0$.

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Require $\Phi(a,\theta) = V(\theta)$:

$$(A_0 + B_0 a^{-1}) + \sum_{\ell>0}^{\infty} B_\ell a^{-(\ell+1)} P_\ell(\cos\theta) = V(\theta)$$

Use $\int_{-1}^{1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$ (Jackson (3.21))

$$\Rightarrow (A_\ell a^\ell + B_\ell a^{-(\ell+1)}) \frac{2}{2\ell+1} = \int_{-1}^{1} P_\ell(x) V(x)\, dx$$

$$= \int_{-\cos\beta}^{\cos\beta} P_\ell(x) V_0\, dx$$

Use $P_\ell(x) = \frac{1}{2\ell+1}\left[P'_{\ell+1}(x) - P'_{\ell-1}(x)\right]$

$$\Rightarrow \int_{-\cos\beta}^{+\cos\beta} P_\ell(x) V_0\, dx = \frac{V_0}{2\ell+1} \int_{-\cos\beta}^{+\cos\beta} [P'_{\ell+1}(x) - P'_{\ell-1}(x)]\, dx$$

$$= \frac{V_0}{2\ell+1}\Big[(P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)) - (P_{\ell+1}(-\cos\beta) - P_{\ell-1}(-\cos\beta)) \Big]$$

(cont'd)

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$$\int_{-\cos\beta}^{+\cos\beta} P_\ell(x) V_0\, dx = \frac{V_0}{2\ell+1}\left(1-(-)^{\ell+1}\right)\left(P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right)$$

$$\Rightarrow A_\ell a^\ell + B_\ell a^{-(\ell+1)} = \frac{V_0}{2}\left(1-(-)^{\ell+1}\right)\left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right]$$

$\underline{\ell=0:}$

$$A_0 + B_0 a^{-1} = \frac{V_0}{2}(2)\left[P_1(\cos\beta) - 0\right]$$

$$= V_0 \cos\beta$$

$\underline{\ell>0:}$

$$B_\ell = \frac{V_0}{2} a^{\ell+1}\left(1-(-)^{\ell+1}\right)\left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right]$$

Require $\Phi(r,\theta) \to 0$ as $r \to \infty$:

$\lim_{r\to\infty} \Phi(r,\theta) = A_0$ hence $A_0 = 0$

Since $1-(-)^{\ell+1} = \begin{cases} 0 & \ell \text{ odd} \\ 2 & \ell \text{ even,} \end{cases}$

we have

$$\boxed{\Phi(r,\theta) = \sum_{\ell=0,2,4,\dots}^{\infty} V_0\left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right]\left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)}$$

(c) (20 points) Far from the origin, the potential will look like it was produced by a set of multipoles. Use your results above to compute the multipole moments $q_{\ell m}$ defined by the potential, for all ℓ and m.

The potential outside a charge distribution is (Jackson (4.1))

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell,m} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta,\phi)$$

Here,

$$\Phi(\vec{x}) = \sum_{\ell=0,2,4\ldots}^{\infty} V_0 \left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right] \left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

so immediately we see $\boxed{q_{\ell m} = 0 \text{ for } m \neq 0 \text{ or } \ell \text{ odd.}}$

For the remaining cases use

$$Y_{\ell 0}(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\right]^{1/2} P_\ell(\cos\theta) \qquad \text{(Jackson (3.57))}$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0} \frac{4\pi}{2\ell+1} q_{\ell 0} \left[\frac{2\ell+1}{4\pi}\right]^{1/2} = V_0\left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right] a^{\ell+1}$$

$$\Rightarrow \boxed{q_{\ell 0} = \epsilon_0 \left[(4\pi)(2\ell+1)\right]^{1/2} V_0 \left[P_{\ell+1}(\cos\beta) - P_{\ell-1}(\cos\beta)\right] a^{\ell+1} \text{ for } \ell \text{ even}}$$

2. (a) (15 points) A function $f(r)$ for $0 < r < \infty$ can be expressed in terms of an analogue of a Fourier transform known as a *Hankel transform*, given by

$$f(r) = \int_0^\infty F(k) J_\mu(kr) k dk, \tag{1}$$

$$F(k) = \int_0^\infty f(r) J_\mu(kr) r dr, \tag{2}$$

for any fixed μ.

Using orthogonality properties of Bessel functions (see e.g. Jackson section 3.8), demonstrate the following relation, known as the Parseval relation:

$$\int_0^\infty (f(r))^2 r dr = \int_0^\infty (F(k))^2 k dk. \tag{3}$$

$$\int_0^\infty (f(r))^2 r dr = \int_0^\infty r dr \int_0^\infty F(k_1) J_\mu(k_1 r) k_1 dk_1 \int_0^\infty F(k_2) J_\mu(k_2 r) k_2 dk_2$$

Use $\int_0^\infty J_\mu(k_1 r) J_\mu(k_2 r) r dr = \frac{\delta(k_1 - k_2)}{k_1}$ $\qquad$ J(3.108)

$$= \int_0^\infty k_1 dk_1 \int_0^\infty k_2 dk_2 \, F(k_1) F(k_2) \frac{\delta(k_1 - k_2)}{k_1}$$

$$= \int_0^\infty k dk \, (F(k))^2$$

(b) (25 points) Suppose an infinite plane, parallel to the xy plane but located at $z = L$, carries an azimuthally-symmetric potential $V(\rho)$. (V does not depend upon ϕ.)

Find an expression for the potential $\Phi(\rho, \phi, z)$ for all points above this plane, meaning for all $z > L$. Assume that $\Phi \to 0$ as $z \to \infty$.

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The general sol'n to Laplace's eq'n can be written

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, e^{-kz} J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right]$$

J(3.106)

Require $\Phi(z=L) = V(\rho)$

so

$$V(\rho) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, e^{-kL} J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right]$$

Since V is independent of ϕ,
from orthogonality of trig functions we have
$A_m(k) = 0$, and $B_m(k) = 0$ for $m \neq 0$

$$\Rightarrow V(\rho) = \int_0^{\infty} dk\, e^{-kL} J_0(k\rho)\, B_0(k)$$

Use the same orthogonality condition as (a):

$$\int_0^{\infty} \rho\, d\rho\, V(\rho) J_0(k\rho) = \int_0^{\infty} \rho\, d\rho\, J_0(k\rho) \int_0^{\infty} d\tilde{k}\, e^{-\tilde{k}L} J_0(\tilde{k}\rho)\, B_0(\tilde{k})$$

$$= \int_0^{\infty} d\tilde{k}\, e^{-\tilde{k}L} B_0(\tilde{k}) \frac{\delta(k-\tilde{k})}{k}$$

(cont'd)

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$$\int_0^\infty \rho\, d\rho\, V(\rho)\, J_0(k\rho) = \frac{e^{-kL}}{k} B_0(k)$$

$$\Rightarrow \boxed{B_0(k) = k e^{kL} \int_0^\infty \rho\, d\rho\, V(\rho)\, J_0(k\rho)}$$

and

$$\boxed{\begin{aligned}\Phi(\rho, \phi, z) &= \int_0^\infty dk\, e^{-kz} J_0(k\rho) B_0(k) \\ &= \int_0^\infty k\, dk\, e^{k(L-z)} J_0(k\rho) \int_0^\infty \rho'\, d\rho'\, V(\rho')\, J_0(k\rho')\end{aligned}}$$