Physics 5406: Classical electromagnetism II

Spring 2024

Test 1

February 19, 2024

NAME: ______________________________

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has ???? pages, including this cover sheet. If it does not, see me.

Solutions

1. (~~???~~ 10 points) Show that in the absence of charge and current sources, conservation of linear momentum of the electromagnetic field is equivalent to the statment

$$\sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta} = 0$$

where $T_{\alpha\beta}$ is the Maxwell stress tensor defined in Jackson (6.120). (Note: a long detailed computation is not required.)

(6.121) says,

$$\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{field}\right)_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} \, d^3x$$

If there are no sources, then $\vec{P}_{mech} = 0$,

so this becomes

$$\frac{d}{dt}\left(\vec{P}_{field}\right)_\alpha = \int_V d^3x \left[\sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\right]$$

& the result follows.

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4. Conservation of angular momentum. The mechanical torque density on a given system of charges and currents is

$$\vec{\tau}_{\rm mech} = \vec{x} \times \left(\rho \vec{E} + \vec{J} \times \vec{B}\right),$$

and the angular momentum density of the electromagnetic field is

$$\vec{L}_{\rm field} = \vec{x} \times \vec{P}_{\rm field} = \vec{x} \times \epsilon_0 \left(\vec{E} \times \vec{B}\right).$$

(a) (10 points) Using results in the text, briefly derive an expression for

$$\vec{\tau}_{\rm mech} + \frac{d}{dt} \vec{L}_{\rm field}$$

in terms of the Maxwell stress tensor $T_{\alpha\beta}$.

$$\vec{\tau}_{mech} = \vec{x} \times \dot{\vec{P}}_{mech}$$

$$\dot{\vec{P}}_{mech} + \dot{\vec{P}}_{field} = \vec{T} \quad \text{for } T_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta}$$

so

$$\vec{\tau}_{mech} + \dot{\vec{L}}_{field} = \vec{x} \times \vec{T}$$

$$\left(\vec{\tau}_{mech} + \dot{\vec{L}}_{field}\right)_\alpha = (\vec{x} \times \vec{T})_\alpha$$

$$= \sum_{\beta,\gamma,\lambda} \epsilon_{\alpha\beta\gamma} x_\beta \frac{\partial}{\partial x_\lambda} T_{\gamma\lambda}$$

$$= \sum_{\beta,\gamma,\lambda} \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\lambda} \left(x_\beta T_{\gamma\lambda}\right)$$

for $\epsilon_{\alpha\beta\gamma}$ = Levi-Civita symbol

$$= \begin{cases} \pm 1 & \alpha\beta\gamma \text{ permutation of } 123 \\ 0 & \text{else} \end{cases}$$

(b) (5 points) In the special case that

$$\sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta} = 0,$$

what can be said about the relation between the external torque $\vec{\tau}_{\text{mech}}$ and the field angular momentum density $\vec{L}_{\text{field}}$?

$$\vec{\tau}_{mech} = -\dot{\vec{L}}_{field}$$

(c) (5 points) Suppose one has a plastic disk that is free to rotate in the xy plane. A thin ring of uniformly distributed charge is embedded in the disk. Perpendicular to the disk is a solenoid carrying a current, hence creating a magnetic field (passing perpendicularly through the disk). If the battery driving the current is disconnected, so that the current dies down, then the magetic field in the solenoid will decrease.

Briefly explain what happens next, in words, and why no conservation laws are violated.

(No computations are expected, just a short explanation.)

From Lenz's law, as the magnetic flux drops,
an electric field arises which torques the ring of charge,
so the disk starts to spin.

The angular momentum comes from the field.

See Feynman lectures §17-4, 27-1

3. Consider waves propagating down a waveguide whose cross-section is a rectangular box of sides of length α, β, with sides parallel to the x, y axes respectively, and the waveguide itself oriented parallel to the z axis.

   (a) (~~???~~ 20 points) Show that it is possible to superimpose a TE_{mn} and TM_{mn} mode, with $m > 0$ and $n > 0$, to get a field with $H_x = 0$.

TE mode: $\psi^{TE}_{mn} = H_0 \cos\left(\frac{mx\pi}{\alpha}\right)\cos\left(\frac{n\pi y}{\beta}\right) = H_z$

$$H_t = \frac{ik}{\gamma^2}\nabla_t \psi^{TE}$$

$$\Rightarrow H_x^{TE} = \frac{-ik}{\gamma^2} H_0 \frac{m\pi}{\alpha}\sin\left(\frac{mx\pi}{\alpha}\right)\cos\left(\frac{n\pi y}{\beta}\right)$$

TM mode: $\psi^{TM}_{mn} = E_0 \sin\left(\frac{mx\pi}{\alpha}\right)\sin\left(\frac{n\pi y}{\beta}\right)$

$$E_t = \frac{ik}{\gamma^2}\nabla_t \psi^{TM}, \quad H_z = \frac{1}{Z}\hat{z}\times E_t$$

$$\Rightarrow H_x^{TM} = \frac{1}{\sqrt{\mu/\varepsilon}}(-E_y)$$

$$= -\frac{1}{\sqrt{\mu/\varepsilon}}\frac{ik}{\gamma^2}E_0\left(\frac{n\pi}{\beta}\right)\sin\left(\frac{mx\pi}{\alpha}\right)\cos\left(\frac{n\pi y}{\beta}\right)$$

For both,

$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{\alpha^2}+\frac{n^2}{\beta^2}\right)$ & both H_x have same functional form.

Sum:

$$H_x = H_x^{TE} + H_x^{TM}$$

$$= \frac{-ik}{\gamma^2}\left(H_0\frac{m\pi}{\alpha} + \sqrt{\frac{\varepsilon}{\mu}}E_0\frac{n\pi}{\beta}\right)\sin\left(\frac{mx\pi}{\alpha}\right)\cos\left(\frac{n\pi y}{\beta}\right)$$

& so we see

$H_x = 0$ if $H_0\frac{m\pi}{\alpha} + \sqrt{\frac{\varepsilon}{\mu}}E_0\frac{n\pi}{\beta} = 0$

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(b) (~~???~~ points) Compute the ratio of the power transmitted by the two modes above.

From (8.51), transmitted power is

$$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\begin{Bmatrix}\epsilon \\ \mu\end{Bmatrix}\int_A |\psi|^2\,da$$

TM (for ϵ), TE (for μ)

$$\text{TE: } P_{TE} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\mu\,|H_0|^2\int_0^\alpha \cos^2\left(\frac{m\pi x}{\alpha}\right)dx\int_0^\beta \cos^2\left(\frac{n\pi y}{\beta}\right)dy$$

$$\text{TM: } P_{TM} = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\epsilon\,|E_0|^2\int_0^\alpha \sin^2\left(\frac{m\pi x}{\alpha}\right)dx\int_0^\beta \sin^2\left(\frac{n\pi y}{\beta}\right)dy$$

but ω_λ's are same, and $\int_0^a \cos^2\left(\frac{m\pi x}{\alpha}\right)dx = \int_0^\alpha \sin^2\left(\frac{m\pi x}{\alpha}\right)dx$

so

$$\frac{P_{TE}}{P_{TM}} = \frac{\mu|H_0|^2}{\epsilon|E_0|^2}$$

Use $H_0 = -\sqrt{\frac{\epsilon}{\mu}}\,E_0\,\frac{n}{\beta}\frac{\alpha}{m}$

$$\frac{P_{TE}}{P_{TM}} = \left|\frac{n\alpha}{m\beta}\right|^2$$

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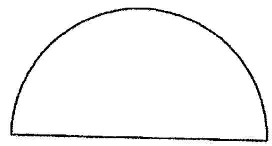
~~3~~. (20 points) A cylindrical waveguide has cross-section given by a half of a circle of radius R, as illustrated below:



Compute E_z for all TM modes propagating along this waveguide. (The next page is left blank as extra space for you to work.)

Solve $(\nabla_t^2 + \gamma^2)\psi = 0$, $\psi = E_z$ (TM),

for $\psi = 0$ at boundary.

Basis for sol'ns that vanish at $\rho = R$:

$$\psi(\rho, \phi) = E_0 J_m(\gamma_{mn}\rho)\, e^{\pm im\phi}$$

where $\gamma_{mn} = \frac{x_{mn}}{R}$, $x_{mn} = n^{th}$ root of J_m

Require $\psi = 0$ for $\phi = 0$:

$$\Rightarrow \psi(\rho, \phi) = E_0 J_m(\gamma_{mn}\rho) \sin m\phi$$

Require $\psi = 0$ for $\phi = \pi$:

$\Rightarrow \sin(m\pi) = 0$, holds automatically.

Basis for solutions:

$$\boxed{E_z = E_0 J_m\left(x_{mn} \frac{\rho}{R}\right) \sin(m\phi)}$$

for m an integer

(This page left blank as extra space for your solution.)

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~~3~~. (15 points) Suppose that

$$\mathrm{Im}\,\epsilon(\omega) = \lambda\epsilon_0\left(\theta(\omega-\omega_1) - \theta(\omega-\omega_2)\right),$$

where $\omega_2 > \omega_1 > 0$ and

$$\theta(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

Compute the real part of $\epsilon(\omega)$. Assume either $\omega > \omega_2$ or $\omega < \omega_1$.

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{2}{\pi} P\int_0^\infty d\omega' \frac{\omega'}{\omega'^2 - \omega^2}\,\mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0} \qquad (7.120)$$

$$= 1 + \frac{2}{\pi}\lambda\, P\int_{\omega_1}^{\omega_2} d\omega' \frac{\omega'}{\omega'^2-\omega^2}$$

$$= 1 + \frac{2}{\pi}\lambda\left(\frac{1}{2}\right)\ln\left(\omega'^2 - \omega^2\right)\Big|_{\omega_1}^{\omega_2}$$

$$= 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right)$$

$\rightsquigarrow$ Kramers - Kronig

There is an alternative expression in terms of an integral from $-\infty$ to $+\infty$, but to use it correctly, you must add negative-frequency contributions so as to maintain reality condition on $\varepsilon(\omega)$, ie, $\varepsilon(-\omega) = \varepsilon(+\omega)^*$.