Physics 5406: Classical electromagnetism II

Spring 2024

Test 2

April 3, 2024

NAME: ______________________________

Instructions:

Do all work to be graded in the space provided. If you need extra space, use the reverse side of a page and indicate on the front that you have continued work on the back. (Otherwise, work on the back of a page is ignored.) Please circle or box or somehow mark your final answers to each question.

Please cross out any work that you do not wish to be considered as part of your solution.

Calculators are NOT allowed on this test.

Please check to be certain that this test has ~~?????~~ 10 pages, including this cover sheet. If it does not, see me.

Solutions

~~5.~~ 1 **Properties of resonant cavity solutions.** Consider a resonant cavity of arbitrary shape.

(a) (15 points) Show that the time-averaged electric and magnetic field energies

$$\frac{\epsilon_0}{4}\int |\vec{E}|^2\, d^3x, \quad \frac{1}{4\mu_0}\int |\vec{B}|^2\, d^3x, \tag{1}$$

are equal. Assume that the wall of the resonant cavity is a perfect conductor. (Hint: Use the wave equation to express $\omega^2 \vec{E}^*$ in terms of the Laplacian of $\vec{E}$, and relate $\nabla \times \vec{E}$ to $\vec{B}$.)

Wave eq'n: $(\nabla^2 + \mu_0\epsilon_0\omega^2)\bar{E} = 0 = (\nabla^2 + \mu_0\epsilon_0\omega^2)\bar{B}$

$$\frac{\mu_0\epsilon_0\omega^2}{4}\int |E|^2 d^3x = -\frac{1}{4}\int \bar{E}^* \cdot \nabla^2 \bar{E}\, d^3x = +\frac{1}{4}\int \bar{E}^* \cdot \nabla\times(\nabla\times E)\, d^3x$$

$$= +\frac{1}{4}\int (\nabla\times E^*)\cdot(\nabla\times\bar{E})\, d^3x$$

after integrate by parts & use $\bar{E}=0$ at boundary

Use $\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t} = +i\omega\bar{B}$

$$= +\frac{1}{4}\omega^2\int |B|^2 d^3x$$

$$\Rightarrow \frac{\epsilon_0}{4}\int |E|^2 d^3x = \frac{1}{4\mu_0}\int |B|^2 d^3x$$

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(b) (~~??~~ points) For two resonant cavity modes of different frequences $\omega_1 \neq \omega_2$, show that

$$\int \vec{E}_1^* \cdot \vec{E}_2 \, d^3x = 0 \cancel{= \int \vec{B}_1^* \cdot \vec{B}_2 d^3x}. \tag{2}$$

(Hint: use the same method as in part (a).)

$$\mu_0 \varepsilon_0 \omega_2^2 \int \bar{E}_1^* \cdot \bar{E}_2 \, d^3x = -\int \bar{E}_1^* \cdot \nabla^2 \bar{E}_2 \, d^3x$$

$$= +\int \bar{E}_1^* \cdot \nabla \times (\nabla \times \bar{E}_2) \, d^3x = +\int (\nabla \times \bar{E}_1^*) \cdot (\nabla \times \bar{E}_2) \, d^3x$$

$$= \omega_1 \omega_2 \int \bar{B}_1^* \cdot \bar{B}_2 \, d^3x$$

$$\mu_0 \varepsilon_0 \omega_1^2 \int \bar{E}_1^* \cdot \bar{E}_2 \, d^3x = \omega_1 \omega_2 \int \bar{B}_1^* \cdot \bar{B}_2 \, d^3x \quad \text{similarly}$$

$$\Rightarrow (\omega_2^2 - \omega_1^2) \int \bar{E}_1^* \cdot \bar{E}_2 \, d^3x = 0$$

so either $\omega_1 = \omega_2$

or $\int \bar{E}_1^* \cdot \bar{E}_2 \, d^3x = 0$

2 15

~~1~~. **Dipole radiation.** (~~??~~ points) A nonrelativistic electron of charge e and mass m is driven by a time-dependent electric field $\vec{E} = E_0\hat{z}\exp(-i\omega t)$, for constant E_0. The resulting radiation is given by that of an oscillating dipole $\vec{p}(t) = e\vec{x}(t)$. Compute $\vec{p}(t)$, and the time-averaged power radiated per solid angle $dP/d\Omega$. (For simplicity, assume that the electron is at rest at the origin at time $t = 0$.) Hint: recall that the force on a charge e is $\vec{F} = e\vec{E}$.

$$\bar{F} = m\ddot{\bar{x}} = e\bar{E} = eE_0\hat{z}e^{-i\omega t}$$

Spose $\bar{x}(0) = 0$, $\dot{\bar{x}}(0) = 0$

$$\Rightarrow \bar{x}(t) = -\frac{eE_0}{m\omega^2}\hat{z}e^{-i\omega t}$$

$$\Rightarrow \boxed{\bar{p}(t) = e\bar{x}(t) = -\frac{e^2E_0}{m\omega^2}\hat{z}e^{-i\omega t}}$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\bar{p}|^2\sin^2\theta$$ Jackson (9.23)

$$= \frac{c^2 Z_0}{32\pi^2}k^4\frac{e^4E_0^2}{m^2\omega^4}\sin^2\theta$$

Use $k = \frac{\omega}{c}$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2c^2}\frac{e^4E_0^2}{m^2}\sin^2\theta}$$

~~2.~~ 3. ~~(?? points)~~ A nonrelativistic electron of charge e and mass m moves in a uniform and constant magnetic field $\vec{B} = -B_0\hat{z}$ for $B_0 > 0$.

(a) ~~(??~~ 5 points) Assume that the electron moves in a circle of radius R. What is the angular frequency ω of its motion, using $\vec{F} = e\vec{v}\times\vec{B}$?

$$m\ddot{\bar{x}} = e\dot{\bar{x}}\times\bar{B}$$

Write $\bar{x} = R\cos\omega t\,\hat{x} + R\sin\omega t\,\hat{y}$

$$\dot{\bar{x}} = -\omega R\sin\omega t\,\hat{x} + \omega R\cos\omega t\,\hat{y}$$

$$\dot{\bar{x}}\times\hat{z} = +\omega R\sin\omega t\,\hat{y} + \omega R\cos\omega t\,\hat{x}$$

$$\ddot{\bar{x}} = -\omega^2(R\cos\omega t\,\hat{x} + R\sin\omega t\,\hat{y})$$

$$m\ddot{\bar{x}} = e\dot{\bar{x}}\times\bar{B}$$

$$\Rightarrow -m\omega^2(R\cos\omega t\,\hat{x} + R\sin\omega t\,\hat{y}) = -eB_0\omega(R\cos\omega t\,\hat{x} + R\sin\omega t\,\hat{y})$$

$$\Rightarrow \boxed{\omega = \frac{eB_0}{m}}$$

(b) (~~22~~ 10 points) Describe the system as a rotating dipole $\vec{p}(t) = e\vec{x}(t)$, and find the time-averaged radiated power per solid angle $dP/d\Omega$.

$$\bar{p}(t) = e\bar{x}(t)$$

$$= eR\cos\omega t\,\hat{x} + eR\sin\omega t\,\hat{y}$$

$$= \mathrm{Re}\left\{ eRe^{-i\omega t}(\hat{x} + i\hat{y}) \right\}$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| eR(\hat{x} + i\hat{y}) \right|^2 \sin^2\theta$$

where θ = angle between $\hat{x} + i\hat{y}$ & vector to observer

$$= \frac{c^2 Z_0}{32\pi^2} k^4 e^2 R^2 (2) \sin^2\theta$$

$$= \boxed{\frac{c^2 Z_0}{16\pi^2} k^4 e^2 R^2 \sin^2\theta}$$

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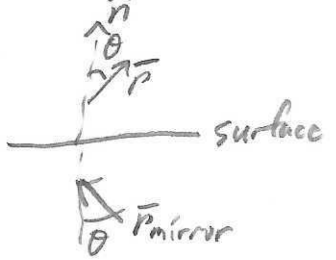
~~6~~. **Dipole radiation near a surface.** (15 ~~22~~ points) Place a harmonically-time-varying dipole, of dipole moment $\vec{p}\exp(-i\omega t)$, near the surface of a perfect conductor, assumed here to be an infinite plane. Assume that the dipole is close enough to the surface that its image oscillates in phase – a reasonable approximation when the distance to the conductor is much less than a single wavelength.

For a dipole $\vec{p}$ making an angle θ with respect to the normal to the surface, compute the total radiated power. At which values of θ is the radiated power enhanced, and at which values of θ is it suppressed?



Using the method of images, we can replace the surface with a 'mirror' dipole $\bar{p}_{mirror}$, as shown.

Total equivalent dipole $\bar{p}_T = \bar{p} + \bar{p}_{mirror}$
$= 2|\bar{p}|\cos\theta\,\hat{n}$

Total power radiated by a dipole $\bar{p}$ is $\dfrac{c^2 Z_0 k^4}{12\pi}|\bar{p}|^2$ (Jackson (9.24))

Here, only see half the total power
(b/c only half volume is on one side of conductor).

$\Rightarrow$ total power radiated
$$= \frac{1}{2}\frac{c^2 Z_0 k^4}{12\pi}|\bar{p}_T|^2 = \frac{1}{2}\frac{c^2 Z_0 k^4}{12\pi}(2)^2|\bar{p}|^2\cos^2\theta$$

$$\boxed{= \frac{c^2 Z_0 k^4}{6\pi}|\bar{p}|^2\cos^2\theta}$$

Maximum when $\theta = 0, \pi$ (parallel, antiparallel $\hat{n}$)

Minimum $(=0)$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $(\perp \hat{n})$

5. (15 points) Suppose some thin bar magnet of length L has harmonically-time-varying magnetization:

$$\vec{\mathcal{M}} = \begin{cases} \mathcal{M}_0 \hat{z} \exp(-i\omega t) & 0 \le z \le L, x = y = 0, \\ 0 & \text{else.} \end{cases} \tag{3}$$

Assume $\rho = 0$ and $\vec{J} = 0$. Compute the coefficients of electric multipole radiation $a_E(\ell, m)$ and magnetic multipole radiation $a_M(\ell, m)$, up to a single one-parameter integral. For which values of ℓ, m are they nonzero?

(The next page is left blank as extra space for you to work.)

From (9.168), since $\rho=0$ & $\vec{J}=0$, & using $\nabla\cdot\vec{m}=0$,

$$a_M(\ell,m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}}\int d^3x \, Y^*_{\ell m}(-k^2)(\vec{r}\cdot\vec{m})\, j_\ell(kr)$$

$$= \frac{-k^4 m_0}{i\sqrt{\ell(\ell+1)}}\int_0^L r^2 dr \int d\Omega \, r\cos\theta \, \delta(\cos\theta - 1)\, Y^*_{\ell m}\, j_\ell(kr)$$

Using $\int_0^{2\pi} d\phi \, e^{im\phi} = 2\pi\delta_{m,0}$

$$= \frac{-k^4 m_0}{i\sqrt{\ell(\ell+1)}}\, Y^*_{\ell m}(\theta=0)(2\pi)\,\delta_{m,0}\int_0^L dr\, r^3 j_\ell(kr)$$

Use $Y_{\ell 0}(\theta,\phi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} P_\ell(\cos\theta)$ (3.57)

$$= -\frac{k^4 m_0}{i}\frac{2\pi}{\sqrt{4\pi}}\left(\frac{2\ell+1}{\ell(\ell+1)}\right)^{1/2}\underbrace{P_\ell(1)}_{=1}\delta_{m,0}\int_0^L dr\, r^3 j_\ell(kr)$$

$$\boxed{a_M(\ell,m) = -\frac{k^4 m_0}{i}\left(\frac{\pi(2\ell+1)}{\ell(\ell+1)}\right)^{1/2}\delta_{m,0}\int_0^L dr\, r^3 j_\ell(kr)}$$

(This page left blank as extra space for your solution.)

From (9.167), since $\rho = 0$ & $\bar{J} = 0$,

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x\, Y_{lm}^* (-ik)\, \nabla \cdot (\bar{r} \times \bar{m})\, j_l(kr)$$

$$\bar{r} \times \hat{z} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ 0 & 0 & 1 \end{vmatrix} = y\hat{x} - x\hat{y}$$

$$\nabla \cdot (\bar{r} \times \hat{z}) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$$

$$\Rightarrow \nabla \cdot (\bar{r} \times \bar{m}) = 0$$

$$\Rightarrow \boxed{a_E(l,m) = 0 \text{ for all } l, m}$$

6. (15 points) Recall that the ordinary electrostatic multiple moment of a system of charges is

$$q_{\ell m} = \int d^3x Y^*_{\ell m} r^\ell \rho(\vec{x}), \tag{4}$$

where ρ is the charge density. Suppose that the charges themselves vary in time so that the multipole moments are functions of t, $q_{\ell m}(t)$, **but not necessarily harmonically-varying** (i.e. with possibly many different frequency components). How would you describe the electric multipole moments $a_E(\ell, m)$ of this system in the long-wavelength limit? Assume that there is no intrinsic magnetization, $\vec{\mathcal{M}} = 0$.

First, take a Fourier transform:

$$q_{\ell m}(t) = \int_{-\infty}^{\infty} d\omega\, \tilde{q}_{\ell m}(\omega) e^{-i\omega t}$$

Then, for any fixed frequency component,

$$\boxed{a_E(\ell, m, \omega) = \frac{ck^{\ell+2}}{i\,(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2} \tilde{q}_{\ell m}(\omega)}$$

(applying Jackson (9.169))

for $k = \omega/c$

Inverting the Fourier transform, we have

$$a_E(\ell, m, t) = \int_{-\infty}^{\infty} \frac{c}{i\,(2\ell+1)!!}\left(\frac{\omega}{c}\right)^{\ell+2}\left(\frac{\ell+1}{\ell}\right)^{1/2} \tilde{q}_{\ell m}(\omega) e^{-i\omega t}$$

$$\neq \frac{ck^{\ell+2}}{i\,(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2} q_{\ell m}(t) \quad \text{in general}$$