Physics 5455 – Problem set 4

1. **Superposition state and time evolution.** Consider a particle of mass m confined to an infinite square well of width a. Suppose the system is initially in the state

$$|\psi, t = 0\rangle \;=\; C\left(|\varphi_1\rangle \;+\; 2|\varphi_3\rangle \;-\; 2|\varphi_4\rangle\right)$$

where the $\{|\varphi_n\rangle\}$ are an orthonormal set of energy eigenstates, $H|\varphi_n\rangle = E_n|\varphi_n\rangle$.

   (a) Determine C and $|\psi, t\rangle$ for $t > 0$.

   (b) Compute the mean energy $\langle H\rangle_\psi$ of this superposition state at time t.

   (c) Compute the Heisenberg picture state $|\psi\rangle_H$ corresponding to the Schrödinger picture state $|\psi, t\rangle$.

   (d) Briefly explain why the result in (b) does not depend upon whether one works in Heisenberg or Schrödinger picture.

2. **Hermite polynomials.** The Hermite polynomials $H_n(x)$ can be defined by a 'generating function' $g(x, t)$, as

$$g(x, t) \;\equiv\; \exp(-t^2 + 2tx) \;=\; \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

By differentiating the generating function with respect to x and t, derive the recurrence relations

$$H_{n+1}(x) \;=\; 2xH_n(x) \;-\; 2nH_{n-1}(x)$$
$$H'_n(x) \;=\; 2nH_{n-1}(x)$$

3. Use the recurrence relations of the last problem to show that the Hermite polynomial $H_n(x)$ satisfies the ODE

$$H''_n(x) \;-\; 2xH'_n(x) \;+\; 2nH_n(x) \;=\; 0$$

4. From the generating function given in a previous problem, derive the orthonormality relation between Hermite polynomials

$$\int_{-\infty}^{\infty} dx H_m(x)H_n(x)e^{-x^2} \;=\; 2^n\sqrt{\pi}\, n!\, \delta_{m,n}$$

Hint: multiply two copies of the generating function and e^{-x^2}, and integrate.