Physics 5455 – Problem set 5

1. Using the recurrence relation for Hermite polynomials, show that
$$a\psi_n(x) \;=\; \sqrt{n}\,\psi_{n-1}(x)$$
for $n > 0$. (Note that for full credit, you should work with wavefunctions, not repeat the ket argument given in class.)

2. Derive the Heisenberg equations of motion for x_H, p_H in the one-dimensional simple harmonic oscillator
$$H \;=\; \frac{p^2}{2m} \;+\; \frac{1}{2}m\omega^2 x^2$$
and compare with the classical equations of motion.

3. **Harmonic oscillator coherent states.**
   (a) Evaluate the inner product $\langle\lambda|\lambda'\rangle$ between two coherent states.
   (b) Show that coherent states are over-complete, meaning in this case that
   $$\int d^2\lambda\,|\lambda\rangle\langle\lambda| \;=\; \pi\cdot 1$$
   Hint: Use polar coordinates $\lambda = \rho e^{i\phi}$.
   (c) Show that for the simple harmonic oscillator in any coherent state $|\lambda\rangle$, that
   $$\Delta x\,\Delta p \;=\; \frac{\hbar}{2}$$
   (saturating the Heisenberg inequality), where $(\Delta x)^2 = \langle(x_H - \langle x_H\rangle_\lambda)^2\rangle_\lambda$, $(\Delta p)^2 = \langle(p_H - \langle p_H\rangle_\lambda)^2\rangle_\lambda$.

4. **Harmonic oscillator variants.** Find the energy eigenvalues and eigenfunctions for the following two variants of the harmonic oscillator Hamiltonian. (Note that you do not need to solve differential equations from scratch, rather you can work with existing solutions.)
   (a) Charged particle in a harmonic oscillator potential and a constant electric field E_0:
   $$V(x) \;=\; \frac{1}{2}m\omega^2x^2 \;-\; qE_0x$$
   (b) Harmonic oscillator on half-space
   $$V(x) \;=\; \left\{ \begin{array}{cc} \frac{1}{2}m\omega^2x^2 & x > 0 \\ \infty & x < 0 \end{array} \right.$$

5. **Anticommuting ladder operators.** Consider ladder operators c, $c^\dagger$ that obey the *anti*commutation relation

$$\{c, c^\dagger\} \equiv cc^\dagger + c^\dagger c = 1$$

In this problem you will study the eigenvalue spectrum of the corresponding number operator $\hat{n} \equiv c^\dagger c$:

$$\hat{n}|n\rangle = n|n\rangle$$

(a) First, express $c|n\rangle$ and $c^\dagger|n\rangle$ in terms of the number eigenstates $|n\rangle$.

(b) Assuming there exists a vacuum state $|0\rangle$ with $c|0\rangle = 0$, which are the only allowed eigenvalues of the non-vanishing eigenstates of $\hat{n}$?

(c) Assuming there exists a vacuum state as above, show that

$$\{c, c\} = 0 = \{c^\dagger, c^\dagger\} \text{ and } \hat{n}^2 = \hat{n}.$$