Physics 5455 – Problem set 6 – Revised Mar 18

1. **Reflection-free potentials.** A quantum particle moves in the potential

$$V(x) \;=\; -\frac{\mu(\mu+1)\hbar^2}{2ma^2[\cosh(x/a)]^2}$$

for $\mu > 0$.

   (a) Find the scattering states ($E \geq 0$) as follows. First, set $\psi_k(x) = e^{ikx}\phi_k(y)$ for $E = \hbar^2k^2/2m$ and $y = x/a$. Then, substitute $z = \tanh y$, and show that the resulting power series terminates if μ is an integer.

   (b) For $\mu = 1, 2$, determine the scattering states ($E \geq 0$) explicitly.

2. **Dirac delta function potential.** Consider the time-independent Schrödinger equation with potential

$$V(x) \;=\; \lambda\delta(x)$$

   (a) Assuming that the wavefunction itself is continuous at $x = 0$, show by integrating the Schrödinger equation that the first derivative has a discontinuity given by

$$\lim_{x\to 0^+}\psi'(x) \;-\; \lim_{x\to 0^-}\psi'(x) \;=\; \frac{2m}{\hbar^2}\lambda\psi(0)$$

   (b) Find the reflection and transmission coefficients for a plane wave scattering state ($E > 0$) incident from the left.

   (c) For $\lambda < 0$, find the normalized bound state wavefunction and its energy eigenvalue ($E < 0$).

   (d) Show that the pole in the (analytic continuation of the) transmission amplitude from part (b), coincides with the bound state energy computed in (c).

3. **The Kronig-Penney model.** In this problem we shall study a simple model of electrons in a one-dimensional crystal, interacting only weakly with the ion cores. Begin with Schrödinger's equation for a particle of mass m in one dimension:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} \;+\; V(x)\psi(x) \;= E\psi(x)$$

   (a) First, show that

$$\frac{p^2}{2m}\varphi(p) \;+\; \int_{-\infty}^{\infty}\frac{dq}{2\pi\hbar}\tilde{V}(p-q)\varphi(q) \;=\; E\varphi(p)$$

   where $\tilde{V}$, φ are the Fourier transforms of V, ψ, respectively.

(b) Assume the potential is periodic with period a, described by delta functions:

$$V(x) = A \sum_{n=-\infty}^{\infty} \delta(x + na)$$

Show that the equation above reduces to

$$\left(\frac{p^2}{2m} - E\right) \varphi(p) + \frac{A}{a} \sum_{n=-\infty}^{\infty} \varphi\left(p + \frac{2\pi\hbar}{a} n\right) = 0$$

You will need to use the identity

$$\sum_{n=-\infty}^{\infty} e^{i(\omega - \omega')n} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - \omega' + 2\pi n)$$

(c) Define

$$f(p) = \sum_{n=-\infty}^{\infty} \varphi\left(p + \frac{2\pi\hbar}{a} n\right)$$

Show that

$$\varphi\left(p + \frac{2\pi\hbar}{a} n\right) = -\frac{2m(A/a) f(p)}{(p + (2\pi\hbar/a)n)^2 - 2mE}$$

(d) Sum over all n to derive the constraint

$$\frac{a}{2mA} = -\sum_{n=-\infty}^{\infty} \left[\left(p + \frac{2\pi\hbar}{a} n\right)^2 - 2mE\right]^{-1}$$

(e) Use the identity

$$\cot x = \sum_{n=-\infty}^{\infty} \frac{1}{x + n\pi}$$

and trigonometric identities to show that the constraint above can be rewritten as

$$\cos(pa/\hbar) = \cos(ka) + \left(\frac{mA}{k\hbar^2}\right) \sin(ka)$$

for

$$k^2 = \frac{2mE}{\hbar^2}$$

Note that the left-hand side can only take values in the range $[-1, 1]$, whereas the right-hand side can take values outside that range. The allowed values of k come in segments known as *bands*, and the gaps between bands are known as *band gaps*.