Physics 5455 – Problem set 9

1. **Landau levels.** In class, we derived the energy eigenvalues of a charged particle in a constant magnetic field $\vec{B} = B\hat{e}_z$. In this problem you will study the wavefunctions.

   (a) From the expression
   $$H \;=\; \frac{1}{2m}\left(\vec{p} \,-\, \frac{e}{c}\vec{A}\right)^2 \;+\; e\Phi$$
   derive the form
   $$H \;=\; -\frac{\hbar^2}{2m}\nabla^2 \;-\; \frac{eB}{2mc}L_z \;+\; \frac{e^2B^2}{8mc^2}(x^2+y^2)$$

   (b) Separate variables, and derive a differential equation for the radial part in cylindrical coordinates. Compare the result to the two-dimensional symmetric harmonic oscillator discussed in class.

2. **Crossed homogeneous electric and magnetic fields.** Consider a charged particle in an electric field $\vec{E} = E_0\hat{e}_x$ and magnetic field $\vec{B} = B\hat{e}_z$, with E_0 and B constant. Take $\vec{A} = Bx\hat{e}_y$.

   Show that $[H, p_y] = 0 = [H, p_z]$. Find the energy eigenstates and eigenvalues for this problem.

   Hint: a suitable separation ansatz leads to a one-dimensional problem; complete the square in x in the ensuing Hamiltonian.

3. **Conserved current for particle in electromagnetic field.** Let a particle with mass m, charge e in an electromagnetic field be described by the wavefunction $\psi(\vec{x})$.

   (a) Define charge and current densities as
   $$\rho(\vec{x}) \;=\; e|\psi|^2$$
   $$\vec{\jmath}(\vec{x}) \;=\; \frac{\hbar e}{2mi}\left(\psi^*\nabla\psi \;-\; \psi\nabla\psi^* \;-\; \frac{2ie}{\hbar c}\vec{A}|\psi|^2\right)$$
   Show that these satisfy the continuity equation
   $$\frac{\partial\rho}{\partial t} \;+\; \nabla\cdot\vec{\jmath} \;=\; 0$$

   (b) Verify that ρ and $\vec{\jmath}$ are invariant under gauge transformations
   $$\Phi \;\mapsto\; \Phi' \;=\; \Phi \;-\; \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \quad \vec{A} \;\mapsto\; \vec{A}' \;=\; \vec{A} \;+\; \nabla\Lambda,$$
   $$\psi \;\mapsto\; \psi' \;=\; \exp\left(\frac{ie}{\hbar c}\Lambda\right)\psi$$

4. **Plane rotator in a magnetic field.** In cylindrical coordinates, the Hamiltonian for a plane rotator about the z axis with fixed distance $\rho = a$ from the origin is

$$H = \frac{L_z^2}{2ma^2} = -\frac{\hbar^2}{2ma^2}\frac{\partial^2}{\partial\phi^2}$$

using the fact that in cylindrical coordinates,

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

(a) Find the energy eigenfunctions and associated energy eigenvalues.

(b) Compute the energy eigenfunctions and eigenvalues in the presence of a magnetic field described by the vector potential

$$\vec{A}(\rho,\phi) = \frac{B}{2}\hat{e}_\phi \left\{ \begin{array}{cc} \rho & \rho \leq b \\ b^2/\rho & \rho > b \end{array} \right\} \equiv A_\phi \hat{e}_\phi$$

with $b < a$. What is the associated magnetic field?