Physics 4674/5674 – Problem set 6

4674 & 5674:

1. Compute the Ricci scalar of the Poincaré plane in two dimensions, defined by the metric
$$ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right)$$
over the region $\{y > 0\}$.

2. Prove the Bianchi identity $\nabla_{[\alpha} R_{\rho\sigma]\mu\nu} = 0$. (Hint: work in locally inertial coordinates.) This is in the text, and we did it in class, but because of its importance, I want you to work through it for yourselves.

3. C 3.7 (Warning: the basic idea is simple but there's a lot of algebra)

4. C 3.10

5674 only:

5. Let ∇ be a covariant derivative (not necessarily metric-compatible or torsion-free) on a manifold M. Let ω be a 1-form on M and let X, Y be vector fields on M. Show that $\tilde{\nabla}$, defined by
$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X$$
is also a covariant derivative on M, in the sense of satisfying the identities
   (a) $\tilde{\nabla}_{fX+gY} Z = f\tilde{\nabla}_X Z + g\tilde{\nabla}_Y Z$ for functions f, g and vector fields X, Y, Z
   (b) $\tilde{\nabla}_X(aY + bZ) = a\tilde{\nabla}_X Y + b\tilde{\nabla}_X Z$ for constants a, b
   (c) $\tilde{\nabla}_X(fY) = f\tilde{\nabla}_X Y + (Xf)Y$

   and also show that $\tilde{\nabla}$ has the same torsion as ∇.

6. Let ∇ be a covariant derivative with nonzero torsion tensor T on some manifold M, and let X, Y be vector fields on M. Show that $\tilde{\nabla}$, defined by
$$\tilde{\nabla}_X Y = \nabla_X Y - T(X, Y)$$
is a covariant derivative on M (in the sense of obeying the identities of the previous problem) with torsion $-T$. Show also that ∇' defined by
$$\nabla'_X Y = \frac{1}{2}\left(\nabla_X Y + \tilde{\nabla}_X Y\right)$$
is a covariant derivative (in the sense of obeying the identities of the previous problem) with vanishing torsion.