Physics 5674: General Relativity

Spring 2012

(Take-home) Test 1

February 15, 2012

NAME: ___________________________________

Instructions:

You may use your book, class notes, old homeworks, and homework solutions. You may consult me if you have questions. You may NOT consult other people or other texts or references.

This test is due at the start of class on Monday February 20.

1. Show that under a general coordinate transformation, for any covector $V_\mu \hat{\theta}^\mu$,

   (a) (5 points) the quantity
   $$\partial_\mu V_\nu$$
   does *not* transform as the components of any kind of tensor, but

   (b) (5 points) the quantity
   $$\partial_\mu V_\nu \, - \, \partial_\nu V_\mu$$
   does transform as the components of an (antisymmetric) (0,2) tensor.

2. (a) (5 points) Show that $A^{[\mu\nu]\rho} B^\sigma_{\mu\nu} = A^{[\mu\nu]\rho} B^\sigma_{[\mu\nu]}$.

   (b) (5 points) Show that if $S_{\mu\nu} = +S_{\nu\mu}$, then $S_{[\mu\nu]} = 0$ and $S_{(\mu\nu)} = S_{\mu\nu}$.

3. (15 points) Consider a boost between special-relativistic coordinate frames x, x' along the x axis defined by
   $$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} \end{aligned}$$
   (in units where $c = 1$). Using the fact that the Maxwell tensor $F_{\mu\nu}$ is a tensor, determine how the electric and magnetic fields transform under this coordinate transformation.

4. We can add a mass to the photon of electromagnetism. Such a field is known as a *Proca field*, and is described by the action
   $$\int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \, + \, \frac{1}{2} m^2 A_\mu A^\mu \right)$$
   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this problem we will assume we are working in inertial coordinates in special relativity, *i.e.* in special coordinates in which the metric is diagonal and has vanishing first derivatives.

   (a) (10 points) Show that the classical equations of motion of the Proca field (*i.e.* Maxwell's equations with a mass) are
   $$\partial_\mu F^{\mu\rho} \, = \, -m^2 A^\rho$$
   by using Hamilton's least-action principle.

   (b) (5 points) By taking another derivative of the equations of motion above, show that when $m^2 \neq 0$,
   $$\partial_\mu A^\mu \, = \, 0$$

(c) (5 points) Show that

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

in this theory.

(d) (10 points) The energy-momentum tensor for the Proca field is given by

$$T^{\mu\nu} = g^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + m^2 \left(A^\mu A^\nu - \frac{1}{2} g^{\mu\nu} A_\rho A^\rho \right)$$

Show that (for constant metric) the energy-momentum tensor above obeys conservation of energy-momentum so long as the equations of motion of the Proca field and the additional condition $\partial_\mu A^\mu = 0$ are obeyed.

5. Determine whether each of the following maps is injective and/or surjective.

   (a) (5 points) $f : \{x > 0 \,|\, x \in \mathbf{R}\} \to \mathbf{R}$ defined by $f(x) = \log x$.

   (b) (5 points) $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^6$.

   (c) (5 points) $h : \mathbf{R} \to [-1, 1]$ defined by $h(x) = \cos x$.

   (d) (5 points) Which (if any) of the maps f, g, h is an isomorphism?

6. (5 points) Can there exist a homeomorphism $\mathbf{R}^2 \to \mathbf{R}^3$? Explain.

7. (10 points) Is the map $f : \mathbf{R} \to \{x > 0 \,|\, x \in \mathbf{R}\}$ given by $f(x) = \exp(x)$, a diffeomorphism? Explain.