Physics 4674/5674G: General Relativity

Spring 2012

(Take-home) Test 2

March 28, 2012

NAME: ____________________________________

Instructions:

You may use your book, class notes, old homeworks, and my homework solutions. You may consult me if you have questions. You may NOT consult other people or other texts or references.

This test is due at the start of class on Monday April 9.

1. The exterior derivative of a p-form

$$\omega \equiv \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

is defined to be the $(p+1)$-form

$$d\omega \equiv \frac{1}{p!}\left(\partial_\nu\omega_{\mu_1\cdots\mu_p}\right)dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

   (a) (10 points) Show that $d^2\omega = 0$ for any ω.

   (b) (10 points) Show that if we replace ordinary derivatives by the metric-compatible torsion-free covariant derivative ∇, then the exterior derivative is unchanged, *i.e.*

$$\frac{1}{p!}\left(\partial_\nu\omega_{\mu_1\cdots\mu_p}\right)dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} = \frac{1}{p!}\left(\nabla_\nu\omega_{\mu_1\cdots\mu_p}\right)dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

2. (10 points) Write the metric on Euclidean 2-space (x,y) in terms of parabolic cylindrical coordinates (u,v):

$$\begin{aligned} x &= \frac{1}{2}\left(u^2 - v^2\right) \\ y &= uv \end{aligned}$$

3. (10 points) For any tensor $S_{\mu\nu}$, define $S \equiv g^{\mu\nu}S_{\mu\nu}$. For any metric-compatible covariant derivative ∇, it is true that

$$g^{\mu\nu}\nabla_\alpha S_{\mu\nu} = \partial_\alpha S$$

simply as a consequence of metric compatibility. Check the expression above explicitly for the Christoffel connection, by expanding out the Christoffel connections. For full (in fact, any) credit, you must expand out connections, not just appeal to metric compatibility.

4. (10 points) For any vector V^μ, show that

$$\partial_\alpha\left(\sqrt{-g}V^\alpha\right) = \sqrt{-g}\,\nabla_\alpha V^\alpha$$

5. (15 points) Show that

$$\left[\nabla_\rho, \nabla_\sigma\right]S^{\mu_1\mu_2} = R^{\mu_1}{}_{\lambda\rho\sigma}S^{\lambda\mu_2} + R^{\mu_2}{}_{\lambda\rho\sigma}S^{\mu_1\lambda}$$

for ∇ the metric-compatible torsion-free connection defined by the Christoffel symbols, and $S^{\mu_1\mu_2}$ any (2,0)-tensor.

6. Consider a two-dimensional metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2$$

(a) (10 points) Solve for the most general function $f(r)$ such that $R_{\mu\nu} = 0$.

(b) (10 points) Consider parallel-transporting a vector V^μ along a path of constant r, in the metric above. Show that the components satisfy

$$\begin{aligned} \partial_t^2 V^t \; - \; \frac{1}{4}(f')^2 V^t \; &= \; 0 \\ \partial_t^2 V^r \; - \; \frac{1}{4}(f')^2 V^r \; &= \; 0 \end{aligned}$$

7. (15 points) In this problem you are going to expand Einstein's equations perturbatively around flat space. Show that if $T_{\mu\nu} = 0$ (so that Einstein's equations are $G_{\mu\nu} = 0$), and if we expand

$$g_{\mu\nu} \; = \; \eta_{\mu\nu} \; + \; h_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric on flat space, and $h_{\mu\nu}$ is a 'small' perturbation, then to leading order in $h_{\mu\nu}$, Einstein's equations become

$$\begin{aligned} &\eta^{\rho\lambda} \left(\partial_\rho \partial_\mu h_{\lambda\nu} \; - \; \partial_\rho \partial_\lambda h_{\nu\mu} \; - \; \partial_\nu \partial_\mu h_{\lambda\rho} \; + \; \partial_\nu \partial_\lambda h_{\rho\mu} \right) \\ &\quad - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\lambda} \eta^{\alpha\beta} \left(\partial_\rho \partial_\alpha h_{\lambda\beta} \; - \; \partial_\rho \partial_\lambda h_{\beta\alpha} \; - \; \partial_\beta \partial_\alpha h_{\lambda\rho} \; + \; \partial_\beta \partial_\lambda h_{\rho\alpha} \right) \; = \; 0 \end{aligned}$$

This second-order PDE for the perturbation $h_{\mu\nu}$ is a kind of wave equation, and its solutions describe gravitational waves.