1.8.18 The velocity of a two-dim'l flow of liquid is given by

$$\bar{V} = \hat{x}\, u(x,y) \;\; - \hat{y}\, v(x,x)$$

If the liquid is incompressible & the flow is irrotational, show that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Irrotational $\Rightarrow \nabla \times \bar{V} = 0$

$$\nabla \times \bar{V} = \hat{z}\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so $\nabla \times \bar{V} = 0 \;\; \Rightarrow \;\; \underline{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

Incompressible $\Rightarrow \nabla \cdot \bar{V} = 0$

$$\Rightarrow \underline{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0}$$

1.9.3 Show that $\nabla \times (\varphi \nabla \varphi) = 0$

$$\nabla \times (\varphi \nabla \varphi) = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi \varphi_x & \varphi \varphi_y & \varphi \varphi_z \end{bmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}(\varphi \varphi_z) - \frac{\partial}{\partial z}(\varphi \varphi_y)\right] - \hat{y}\left[\frac{\partial}{\partial x}(\varphi \varphi_z) - \frac{\partial}{\partial z}(\varphi \varphi_x)\right]$$

$$+ \hat{z}\left[\frac{\partial}{\partial x}(\varphi \varphi_y) - \frac{\partial}{\partial y}(\varphi \varphi_x)\right]$$

$\hat{x}$ component:

$$\frac{\partial}{\partial y}(\varphi \varphi_z) - \frac{\partial}{\partial z}(\varphi \varphi_y) = (\cancel{\varphi_y \varphi_z} + \cancel{\varphi \varphi_{zy}}) - (\cancel{\varphi_z \varphi_y} + \cancel{\varphi \varphi_{yz}})$$

$$= 0$$

& sim'ly for other component

$\Rightarrow \underline{\nabla \times (\varphi \nabla \varphi) = 0}$

1.9.12 Show that any solution of the equation

$$\nabla\times(\nabla\times\bar{A}) - k^2\bar{A} = 0$$

automatically satisfies the vector Helmholtz equ'n $\nabla^2\bar{A} + k^2\bar{A} = 0$

& the solenoidal condition $\nabla\cdot\bar{A} = 0$.

$\nabla\cdot\nabla\times\bar{V} = 0$ for any $\bar{V}$ $\Rightarrow$ $\nabla\cdot\nabla\times(\nabla\times\bar{A}) = 0$

so $\nabla\cdot(\nabla\times\nabla\times\bar{A} - k^2\bar{A}) = -k^2\nabla\cdot\bar{A} \qquad = 0$

$\Rightarrow \underline{\nabla\cdot\bar{A} = 0}$

For the rest, use the identity (A&W (1.85)):

$$\nabla\times(\nabla\times\bar{V}) = \nabla\nabla\cdot\bar{V} - \nabla^2\bar{V}$$

so $\nabla\times(\nabla\times\bar{A}) - k^2\bar{A} = 0$

$\Rightarrow \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} - k^2\bar{A} = 0$

but we know $\nabla\cdot\bar{A} = 0 \quad \Rightarrow \quad \underline{\nabla^2\bar{A} + k^2\bar{A} = 0}$

1\. $\int_C (x^3+y)\,ds, \qquad C: \; x=3t, \quad y=t^3, \quad t\in[0,1]$

$$= \int_0^1 [(3t)^3+t^3][(3)^2+(3t^2)^2]^{1/2}\,dt$$

$$= (28)\int_0^1 t^3\,(3)\,(1+t^4)^{1/2}\,dt$$

$$= (\cancel{3})(28)\left(\frac{1}{\cancel{4}_2}\right)\left(\frac{2}{\cancel{3}}\right)(1+t^4)^{3/2}\Big|_0^1 = \underline{(14)[2^{3/2}-1]}$$

2\. $\int_C (\sin x+\cos y)\,ds, \qquad$ C is the line segment from $(0,0)$ to $(\pi, 2\pi)$

$\Rightarrow \quad y=2x, \quad x\in[0,\pi]$

$$= \int_0^\pi (\sin x+\cos 2x)\,[1+(2)^2]^{1/2}\,dx$$

$$= \sqrt{5}\left[-\cos x+\frac{1}{2}\sin 2x\right]_0^\pi = \sqrt{5}\,[(-)(-1-1)+0]$$

$$= \underline{2\sqrt{5}}$$

10. $\int_C (y\,dx + x\,dy)$, C is the curve $y = x^2$, $x \in [0,1]$

$$= \int_0^1 \left[(x^2)\,dx + x\,(2x\,dx) \right] = \int_0^1 \left[x^2 + 2x^2 \right] dx$$

$$= x^3 \Big|_0^1 = \underline{\underline{1}}$$

---

6. $\int_C \left[xz\,dx + (y+z)\,dy + x\,dz \right]$; C is the curve $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $t \in [0,1]$

$$= \int_0^1 \left[e^t e^{2t} e^t dt + (e^{-t} + e^{2t})(-e^{-t} dt) + e^t (2e^{2t} dt) \right]$$

$$= \int_0^1 \left[e^{4t} - e^{-2t} - e^t + 2e^{3t} \right] dt$$

$$= \left[\frac{1}{4} e^{4t} + \frac{1}{2} e^{-2t} - e^t + \frac{2}{3} e^{3t} \right]_0^1$$

$$= \underline{\left[\frac{1}{4}(e^4 - 1) + \frac{1}{2}(e^{-2} - 1) - (e - 1) + \frac{2}{3}(e^3 - 1) \right]}$$

9. $\vec{F}(x,y) = -e^{-x}\ln y\,\hat{i} + e^{-x}y^{-1}\hat{j}$

$\frac{\partial}{\partial y}\left(-e^{-x}\ln y\right) = -e^{-x}\left(\frac{1}{y}\right)$

$\frac{\partial}{\partial x}\left(e^{-x}y^{-1}\right) = -e^{-x}\left(\frac{1}{y}\right)$ ✓ conservative

$f(x,y) = \int dx\left[-e^{-x}\ln y\right] = e^{-x}\ln y + c_1(y)$

$\frac{\partial f}{\partial y} = e^{-x}\left(\frac{1}{y}\right) + c_1'(y)$

also $= e^{-x}\left(\frac{1}{y}\right) \Rightarrow c_1'(y) = 0 \Rightarrow c_1(y) = \underline{\text{const}}$

$\Rightarrow \underline{f(x,y) = e^{-x}\ln y + (\text{constant})}$

~~10. $\vec{F}(x,y,z) =$~~

10. $F(x,y,z) = 3x^2\hat{\imath} + 6y^2\hat{\jmath} + 9z^2\hat{k}$

$\frac{\partial}{\partial y}(3x^2) = 0$, $\frac{\partial}{\partial x}(6y^2) = 0$

$\frac{\partial}{\partial z}(3x^2) = 0$, $\frac{\partial}{\partial x}(9z^2) = 0$

$\frac{\partial}{\partial z}(6y^2) = 0$, $\frac{\partial}{\partial y}(9z^2) = 0$ $\Rightarrow$ conservative

$f(x,y,z) = \int dx\,(3x^2) = x^3 + C_1(y,z)$

$\frac{\partial}{\partial y} f(x,y,z) = 0 + \frac{\partial}{\partial y} C_1$
also $= 6y^2$ $\Rightarrow \frac{\partial}{\partial y} C_1 = 6y^2 \Rightarrow C_1 = 2y^3 + C_2(z)$

so far: $f(x,y,z) = x^3 + 2y^3 + C_2(z)$

$\frac{\partial f}{\partial z} = 9z^2$ also $= C_2'(z) \Rightarrow C_2 = 3z^3 + \underline{\text{const}}$

$\Rightarrow f(x,y,z) = x^3 + 2y^3 + 3z^3 + (\text{constant})$

Show that the given line integral is independent of path & then evaluate the integral.

11. $\int_{(-1,2)}^{(3,1)} \left[(y^2+2xy)\,dx + (x^2+2xy)\,dy \right]$

$\frac{\partial}{\partial y}(y^2+2xy) = 2y+2x$

$\frac{\partial}{\partial x}(x^2+2xy) = 2x+2y$ ✓ $\Rightarrow$ conservative, hence path independent

Find f:

$f(x,y) = \int dx \left[y^2+2xy \right] = xy^2 + x^2y + c_1(y)$

$\frac{\partial f}{\partial y} = 2xy + x^2 + c_1'$

also $= x^2+2xy$ $\Rightarrow c_1' = 0$

$\Rightarrow f(x,y) = xy^2 + x^2y + (\text{constant})$

Line integral $= f(3,1) - f(-1,2)$

$= \left[(3)(1)^2 + (3)^2(1) + \cancel{c}\right] - \left[(-)(2)^2 + (-)^2(2) + \cancel{c}\right]$

$= 12 - (-2) = \underline{\underline{14}}$

Q. Show that if $F(x,y,z) = g(x^2+y^2+z^2)(x\bar{i}+y\bar{j}+z\bar{k})$
then F is conservative.
Hint: show that $F = \nabla f$, $f(x,y,z) = \frac{1}{2}h(x^2+y^2+z^2)$, $h(u) = \int g(u)\,du$

Follow the hint.

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2x)\,h'(x^2+y^2+z^2) = x\,g(x^2+y^2+z^2) \checkmark$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(2y)\,h'(x^2+y^2+z^2) = y\,g(x^2+y^2+z^2) \checkmark$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}(2z)\,h'(x^2+y^2+z^2) = z\,g(x^2+y^2+z^2) \checkmark$$
