50 pts ~~total~~

1. Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

$(1,0,0)$, $(0,1,0)$, $(1,1,0)$

5 pts

Let V be the vector space of all 2×2 matrices over the field F.
Show V has dimension 4 by exhibiting a basis with 4 elements.

I claim the following is a basis:

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Check spans:

$$\begin{bmatrix}a&b\\c&d\end{bmatrix} = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Check LI:

$$c_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&1\\0&0\end{bmatrix} + c_3\begin{bmatrix}0&0\\1&0\end{bmatrix} + c_4\begin{bmatrix}0&0\\0&1\end{bmatrix} = 0$$

$$\Rightarrow \underline{c_1 = c_2 = c_3 = c_4 = 0}$$

Let V be a vector space over a subfield F of the complex numbers.
S'pose α, β, γ are linearly independent vectors in V.
Show that $(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are linearly independent.

S'pose $c_1(\alpha+\beta) + c_2(\beta+\gamma) + c_3(\gamma+\alpha) = 0$.

Claim $c_1 = c_2 = c_3 = 0$.

Well, rearrange:

$$\alpha(c_1+c_3) + \beta(c_1+c_2) + \gamma(c_2+c_3) = 0$$

Since α, β, γ are linearly independent, that means

$$c_1+c_3 = 0, \quad c_1+c_2 = 0, \quad c_2+c_3 = 0$$

$$\Rightarrow c_1 = -c_3, \quad c_1 = -c_2, \quad c_2 = -c_3$$

$$\Rightarrow c_2 = c_3, \quad c_2 = -c_3 \quad \Rightarrow \underline{c_2 = c_3 = 0 = c_1}$$

$\Rightarrow (\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are linearly independent

(§ 2.4)

Show that the vectors

$\alpha_1 = (1,1,0,0) \qquad \alpha_2 = (0,0,1,1)$

$\alpha_3 = (1,0,0,4) \qquad \alpha_4 = (0,0,0,2)$

form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

First, show $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis.

LI: Suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$

$\Rightarrow \quad c_1 + c_3 = 0, \quad c_1 = 0, \quad c_2 = 0, \quad c_2 + 4c_3 + 2c_4 = 0$

(reading off the components)

$\Rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0 \qquad \Rightarrow \underline{LI}$

Spans: For any $\alpha = (x, y, z, w)$,
I claim there exists c_1, c_2, c_3, c_4 s.t.
$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$.

Solve: $c_1 + c_3 = x, \quad c_1 = y, \quad c_2 = z, \quad c_2 + 4c_3 + 2c_4 = w$

$\Rightarrow \quad c_1 = y, \quad c_2 = z, \quad c_3 = x - y,$

$c_4 = \frac{1}{2}w - \frac{4}{2}(x-y) - \frac{1}{2}z = \frac{1}{2}w - 2x + 2y - \frac{1}{2}z$

$\Rightarrow$ <u>spans</u>

<u>so</u> $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis.

Plugging in, we find

$(1,0,0,0) = (1,0,0,4) - 2(0,0,0,2) = \alpha_3 - 2\alpha_4$

$(0,1,0,0) = (1,1,0,0) - (1,0,0,4) + 2(0,0,0,2) = \alpha_1 - \alpha_3 + 2\alpha_4$

$(0,0,1,0) = (0,0,1,1) - \frac{1}{2}(0,0,0,2) = \alpha_2 - \frac{1}{2}\alpha_4$

$(0,0,0,1) = \frac{1}{2}(0,0,0,2) = \frac{1}{2}\alpha_4$

(§2.4)

$= \{\alpha_1, \alpha_2, \alpha_3\}$

Let $\mathcal{B}$ be the ordered basis for $\mathbb{R}^3$ consisting of

$\alpha_1 = (1,0,-1), \quad \alpha_2 = (1,1,1) \quad \alpha_3 = (1,0,0)$

What are the coordinates of the vector (a,b,c) in the ordered basis $\mathcal{B}$?

Solve:

$$(a,b,c) = x_1\,(1,0,-1) + x_2\,(1,1,1) + x_3\,(1,0,0)$$

$$\Rightarrow \quad \begin{aligned} a &= x_1 + x_2 + x_3 \\ b &= x_2 \\ c &= -x_1 + x_2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= x_2 - c = \underline{b-c} \\ x_2 &= \underline{b} \\ x_3 &= a - x_1 - x_2 = a - (b-c) - b = \underline{a-2b+c} \end{aligned}$$

(§ 2.4)

Let V be the vector space over the complex numbers of all functions $\mathbb{R} \to \mathbb{C}$, ie, the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

a) Show that f_1, f_2, f_3 are linearly independent.

S'pose $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$.

At $x = 0$, $\Rightarrow$ $\underline{c_1 + c_2 + c_3 = 0}$

at $x = 1$, $\Rightarrow$ $c_1 + e^{i} c_2 + e^{-i} c_3 = 0$

at $x = -1$, $\Rightarrow$ $c_1 + e^{-i} c_2 + e^{+i} c_3 = 0$

$\boxed{\text{Better:}}$

at $x = \pi$, $\Rightarrow$ $c_1 + e^{\pi i} c_2 + e^{-\pi i} c_3 = 0$

$\Rightarrow$ $\underline{c_1 - c_2 - c_3 = 0}$

At $x = \frac{\pi}{2}$, $\Rightarrow$ $c_1 + e^{i\pi/2} c_2 + e^{-i\pi/2} c_3 = 0$

$\Rightarrow$ $\underline{c_1 + ic_2 - ic_3 = 0}$

$$\left.\begin{array}{l} c_1 + c_2 + c_3 = 0 \\ c_1 - c_2 - c_3 = 0 \end{array}\right\} \Rightarrow c_1 = 0$$

$$\left.\begin{array}{l} c_2 + c_3 = 0 \\ c_2 - c_3 = 0 \end{array}\right\} \Rightarrow c_2 = c_3 = 0$$

$\Rightarrow$ $\underline{f_1, f_2, f_3 \text{ are LI}}$

b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$
Find an invertible 3×3 matrix P s.t.

$$g_j = \sum_i P_{ij} f_i$$

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix})$$
$$\sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix})$$

So

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2i & -1/2i \end{bmatrix}}_{P} \begin{bmatrix} 1 \\ e^{ix} \\ e^{-ix} \end{bmatrix} = \begin{bmatrix} 1 \\ \tfrac{1}{2}(e^{ix} + e^{-ix}) \\ \tfrac{1}{2i}(e^{ix} - e^{-ix}) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos x \\ \sin x \end{bmatrix}$$

Note $\det P = -\frac{1}{2i} \neq 0$, so P is invertible.

(§ 2.4)

Let V be the real vector space of all polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree 2 or less, ie, the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x+t, \quad g_3(x) = (x+t)^2$$

Show that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V.
If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of f in this ordered basis $\mathcal{B}$?

Check $\mathcal{B}$ is a basis.

LI: S'pose $c_1 g_1 + c_2 g_2 + c_3 g_3 = 0$. Claim $c_1 = c_2 = c_3 = 0$.

$$\Rightarrow c_1 + c_2(x+t) + c_3(x+t)^2 = 0$$
$$\Rightarrow x^2(c_3) + x(2t c_3 + c_2) + 1(c_1 + t c_2 + t^2 c_3) = 0$$

$$\Rightarrow c_3 = 0, \quad 2t c_3 + c_2 = 0, \Rightarrow c_2 = 0$$
$$c_1 + t c_2 + t^2 c_3 = 0 \Rightarrow c_1 = 0 \qquad \Rightarrow \underline{LI}$$

Spans: Let $f(x) = c_0 + c_1 x + c_2 x^2$.
Claim $f(x) = x_1 g_1 + x_2 g_2 + x_3 g_3$ for some x_1, x_2, x_3.

$$\Rightarrow c_0 + c_1 x + c_2 x^2$$
$$= x_1 + x_2(x+t) + x_3(x+t)^2$$

$\Rightarrow$ comparing coefficient of x,
$$x_3 = c_2, \quad x_2 + 2t x_3 = c_1, \quad x_1 + t x_2 + t^2 x_3 = c_0$$

$$\Rightarrow x_3 = c_2, \quad x_2 = c_1 - 2t c_2, \quad x_1 = c_0 - t(c_1 - 2t c_2) - t^2 c_2$$
$$= c_0 - t c_1 + t^2 c_2$$

$\Rightarrow$ Spans, hence basis, & we also have coordinates.

(§ 2.5)

Let $\alpha_1 = (1,1,-2,1)$, $\alpha_2 = (3,0,4,-1)$, $\alpha_3 = (-1,2,5,2)$

Let $\alpha = (4,-5,9,-7)$, $\beta = (3,1,-4,4)$, $\gamma = (-1,1,0,1)$

Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by the α_i ?

α: S'pose $\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$
Solve for c_1, c_2, c_3:

$$\begin{cases} 4 = c_1 + 3c_2 - c_3 \\ -5 = c_1 + 2c_3 \\ 9 = -2c_1 + 4c_2 + 5c_3 \\ -7 = c_1 - c_2 + 2c_3 \end{cases} \Rightarrow \begin{array}{r} c_1 + 2c_3 = -5 \\ 3(c_2 - c_3) = 9 \\ 4c_2 + 9c_3 = -1 \\ -c_2 = -2 \end{array}$$

$\Rightarrow c_2 = 2$, $c_3 = c_2 - 3 = -1$, $c_1 = -5 - 2c_3 = -3$,
so α is in the subspace of $\mathbb{R}^4$ spanned by the α_i.

β: S'pose $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$
Solve for c_1, c_2, c_3:

$$\begin{cases} c_1 + 3c_2 - c_3 = 3 \\ c_1 + 2c_3 = 1 \\ -2c_1 + 4c_2 + 5c_3 = -4 \\ c_1 - c_2 + 2c_3 = 4 \end{cases} \Rightarrow \begin{array}{r} c_1 + 2c_3 = 1 \\ c_2 - c_3 = 2 \\ 4c_2 + 9c_3 = -2 \\ -c_2 = 3 \end{array}$$

$\Rightarrow c_2 = -3$, $c_3 = c_2 - 2 = -5$, $c_1 = 1 - 2c_3 = 11$
$4c_2 + 9c_3 = -12 - 45 \neq -2$

so β is not in the subspace of $\mathbb{R}^4$ spanned by the α_i.

(cont'd)

(cont'd)

γ: S'pose $\gamma = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$.
Solve for c_1, c_2, c_3:

$$\begin{cases} c_1 + 3c_2 - c_3 = -1 \\ c_1 + 2c_3 = 1 \\ -2c_1 + 4c_2 + 5c_3 = 0 \\ c_1 - c_2 + 2c_3 = 1 \end{cases} \Rightarrow \begin{aligned} c_1 + 2c_3 &= 1 \\ c_2 - c_3 &= -2 \\ 4c_2 + 9c_3 &= 2 \\ -c_2 &= 0 \end{aligned}$$

$$\Rightarrow c_2 = 0,\ c_3 = c_2 + 2 = 2,\ c_1 = 1 - 2c_3 = -3$$

$$\& \ 4c_2 + 9c_3 = 0 + 9(2) = 18 \neq 2$$

so γ is not in the subspace of $\mathbb{R}^4$ spanned by the α_i.

(§ 3.1)

Ex. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional vector space V.

Zero transformation:

Range $= \{0\}$

rank $=$ dim range $= 0$

null space $= V$

nullity $=$ dim null space $=$ dim V

Identity transformation:

Range $= V$

rank $=$ dim range $=$ dim V

null space $= \{0\}$

nullity $=$ dim null space $= 0$

(§ 3.1)

Describe the range and null space of the differentiation transformation on the vector space of polynomials.

$$f(x) = c_0 + c_1 x + \cdots + c_k x^k$$

$$(Df)(x) = c_1 + 2c_2 x + \cdots + k c_k x^{k-1}$$

Range : all polynomials
Null space : constant polynomials $f(x) = c_0$

(§ 3.1)

4. Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-1,1) = (1,0)$, $T(1,1,1) = (0,1)$?

Yes

For example,

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

More generally, to uniquely specify T, I'd need to specify how it acts on 3 LI vectors, but here its action on only 2 has been specified, so there should be more than one linear transformation with the desired property.

(§ 3.1)

Eg. Let V be the vector space of all $n \times n$ matrices over a field F, & let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation $V \to V$.

$$T(cD+E) = (cD+E)B - B(cD+E)$$

$$= c(DB-BD) + (EB-BE)$$

$$= cT(D) + T(E)$$

(§3.1)

Let V be the set of all complex numbers regarded as a vector space over the field of real numbers. Find a function $V \to V$ which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbb{C}$, i.e., which is not complex linear.

Consider $T: z \mapsto \bar{z}$ where $z \in \mathbb{C}$

If we write $z = x + iy$,

then $T: (x, y) \mapsto (x, -y)$

Check linearity over $\mathbb{R}$:

$$
\begin{aligned}
T\big(c(x_1, y_1) + (x_2, y_2)\big) &= T\big((cx_1 + x_2,\ cy_1 + y_2)\big) \\
&= (cx_1 + x_2,\ -cy_1 - y_2) \\
&= c(x_1, -y_1) + (x_2, -y_2) \\
&= cT(x_1, y_1) + T(x_2, y_2)
\end{aligned}
$$

But if c is a complex number,

$$
\begin{aligned}
T(cz_1 + z_2) &= \bar{c}\bar{z}_1 + \bar{z}_2 \\
&\neq c\bar{z}_1 + \bar{z}_2 = cT(z_1) + T(z_2).
\end{aligned}
$$

(§3.1)

1. Let V be a vector space over the field F, n-dim'l
T a linear transformation $V \to V$ s.t. ~~rank~~ ~~range~~ range T = ~~nullity~~ ~~null space~~ null space of T.
Show that n is even.

$$\dim \text{range}(T) + \dim \text{null space}(T) = \dim V = n$$

Here, $LHS = 2 \dim \text{range}(T)$

$$\Rightarrow n = 2 \dim \text{range}(T)$$

$$\Rightarrow n \text{ even.}$$

(§ 3.1)

4. Let V be a vector space, $T: V \to V$ a linear Transformation.
Show that the following statements are equivalent:

- the intersection of the range of T & the null space of T is the zero subspace of V
- if $T(T\alpha) = 0$ then $T\alpha = 0$

S'pose (range T) $\cap$ (null space of T) $= \{0\}$.

Then, s'pose $T(T\alpha) = 0$
$\Rightarrow$ $T\alpha \in$ null space of T
but clearly also $T\alpha \in$ range of T
$\Rightarrow$ $T\alpha \in$ (range) $\cap$ (null space) $= \{0\}$

$\Rightarrow$ $T\alpha = 0$.

S'pose if $T(T\alpha) = 0$ then $T\alpha = 0$.

If $T(T\alpha) = 0$ then $T\alpha \in$ (range) $\cap$ (null space)
so if all such $T\alpha = 0$,
then (range) $\cap$ (null space) $= \{0\}$.