6. A-W 3.2.32

10 pts If A^{-1} has elements

$$(A^{-1})_{ij} = \frac{1}{\det A} \underbrace{(-)^{i+j} M_{ji}}_{\text{cofactor}}$$

then show that $A^{-1}A = 1$.

$$(A^{-1}A)_{ik} = \sum_j (A^{-1})_{ij} A_{jk}$$

$$= \frac{1}{\det A} \sum_j (-)^{i+j} M_{ji} A_{jk}$$

If $k=i$, then $\sum_j (-)^{i+j} M_{ji} A_{ji} = \det A$, expanded along i^{th} column.

If $k \neq i$, then $\sum_j (-)^{i+j} M_{ji} A_{jk}$ = det of a matrix w/ repeated column, hence $= 0$

$$\Rightarrow \sum_j (-)^{i+j} M_{ji} A_{jk} = \delta_{ik} \det A$$

$$\Rightarrow (A^{-1}A)_{ik} = \frac{1}{\cancel{\det A}} \delta_{ik} \cancel{\det A} = \delta_{ik} \quad ✓$$

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7. A-W 3.2.33

5 pts Show that $\det(A^{-1}) = (\det A)^{-1}$

$$A^{-1}A = I \quad \Rightarrow \underset{\det(A^{-1})\det A}{\overset{\shortparallel}{\det(A^{-1}A)}} = \det(I) = 1$$

so $\det(A^{-1})\det A = 1$

$\Rightarrow$ $\underline{\det(A^{-1}) = (\det A)^{-1}}$

1. Let V be the vector space of polynomials of degree ≤ 2, ie, $c_0 + c_1 x + c_2 x^2$

10 pts

with dot product

$$f(x) \cdot g(x) = \int_0^1 f(x) g(x)\, dx$$

Use the Gram-Schmidt procedure to construct an orthonormal basis from the LI vectors

$$\{\underset{\alpha_1}{\overset{\shortparallel}{1}},\ \underset{\alpha_2}{\overset{\shortparallel}{x+1}},\ \underset{\alpha_3}{\overset{\shortparallel}{x^2+x}}\},$$

$w_1 = 1, \quad |w_1|^2 = \int_0^1 (1)^2 dx = x\big|_0^1 = 1 \quad \Rightarrow \boxed{v_1 = 1}$

$w_2 = \alpha_2 - (v_1 \cdot \alpha_2) v_1$:

$$v_1 \cdot \alpha_2 = \int_0^1 (1)(x+1)\,dx = \left[\tfrac{1}{2}x^2 + x\right]_0^1 = \tfrac{1}{2} + 1 = 3/2$$

$$\Rightarrow w_2 = x + 1 - \left(\tfrac{3}{2}\right)(1) = x - \tfrac{1}{2}$$

$$|w_2|^2 = \int_0^1 \left(x - \tfrac{1}{2}\right)^2 dx = \int_0^1 \left(x^2 - x + \tfrac{1}{4}\right) dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{x}{4}\right]_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{4}{12} - \tfrac{6}{12} + \tfrac{3}{12} = \tfrac{1}{12}$$

$$\Rightarrow \boxed{v_2 = \sqrt{12}\left(x - \tfrac{1}{2}\right)}$$

$w_3 = \alpha_3 - (v_1 \cdot \alpha_3) v_1 - (v_2 \cdot \alpha_3) v_2$

$$(v_1 \cdot \alpha_3) = \int_0^1 (1)(x^2 + x)\,dx = \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 = \tfrac{5}{6}$$

$$(v_2 \cdot \alpha_3) = \int_0^1 \sqrt{12}\left(x - \tfrac{1}{2}\right)(x^2 + x)\,dx = \sqrt{12}\int_0^1 \left(x^3 + x^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}x\right) dx$$

$$= \sqrt{12}\left(\cancel{\tfrac{1}{4}} + \tfrac{1}{3} - \tfrac{1}{2}\tfrac{1}{3} - \cancel{\tfrac{1}{2}\tfrac{1}{2}}\right) = \sqrt{12}\,(1/6)$$

$$\Rightarrow w_3 = (x^2 + x) - \left(\tfrac{5}{6}\right)(1) - \left[\sqrt{12}\,(1/6)\right]\left[\sqrt{12}\left(x - \tfrac{1}{2}\right)\right]$$

$$= x^2 + x - \tfrac{5}{6} - 2x + 1 \quad = x^2 - x + 1/6$$

(cont'd)

(cont'd)

$$|w_3|^2 = \int_0^1 (x^2 - x + 1/6)^2 dx$$

$$= \int_0^1 \left[x^4 - x^3 + \tfrac{1}{6}x^2 - x^3 + x^2 - \tfrac{1}{6}x + \tfrac{1}{6}x^2 - \tfrac{1}{6}x + \tfrac{1}{36}\right] dx$$

$$= \int_0^1 \left[x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36}\right] dx$$

$$= \frac{1}{5} - \frac{2}{4} + \frac{4}{3}\frac{1}{3} - \frac{1}{3}\frac{1}{2} + \frac{1}{36} = \frac{1}{5} - \frac{1}{2} + \frac{1}{3}\underbrace{\left(\frac{4}{3} - \frac{1}{2}\right)}_{5/6} + \frac{1}{36}$$

$$= \frac{1}{5} - \frac{18}{36} + \frac{10}{36} + \frac{1}{36} = \frac{1}{5} - \frac{7}{36}$$

$$= \frac{36}{180} - \frac{35}{180} = \frac{1}{180}$$

$$\Rightarrow \boxed{v_3 = \sqrt{180}\,(x^2 - x + 1/6)}$$

$\{v_1, v_2, v_3\}$ is an orthonormal basis