(A-W 5.2.6(a-c))

Test for convergence

a) $\sum_{n=2}^{\infty} (\ln n)^{-1}$

For $n \geq 2$, $\ln n \leq n \Rightarrow (\ln n)^{-1} \geq n^{-1}$

Apply Comparison test: $\sum n^{-1}$ diverges, hence $\sum (\ln n)^{-1}$ diverges.

b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

Apply ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} = \frac{n+1}{10}$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to \infty > 1$ hence diverges

c) $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}$

Comparison test: $\frac{1}{2n(2n+1)} \leq \frac{1}{(2n)^2} = \frac{1}{4n^2}$

$\sum \frac{1}{4n^2}$ converges, so this series must also converge

(A-W 5.2.13)

(Olbers paradox) Assume a static universe in which the stars are uniformly distributed, divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of ω_0. Allowing for the blocking out of distant stars by nearer stars, show that the total net solid angle subtended by all stars, shells extending to infinity, is exactly 4π. (Therefore the night sky should be ablaze with light.)

1st shell: stars cover ω_0

2nd shell: stars cover an additional $\omega_0 - \left(\frac{\omega_0}{4\pi}\right)\omega_0$, subtracting out the contribution of those covered by nearer star

3rd shell: stars cover an additional $\omega_0\left(1-\frac{\omega_0}{4\pi}\right)^2$, subtracting out the contribution of those covered by nearer stars

...

Total solid angle subtended $= \sum_{n=0}^{\infty} \omega_0\left(1-\frac{\omega_0}{4\pi}\right)^n$

→ geometric series

→ sums to $\dfrac{\omega_0}{1-\left(1-\frac{\omega_0}{4\pi}\right)} = \dfrac{\omega_0}{\omega_0/4\pi} = \underline{4\pi}$ ✓

(AW 5.4.3a)

Show that $\displaystyle\sum_{n=2}^{\infty} [\zeta(n) - 1] = 1$

~~10 pts~~ 10 pts

where $\zeta(n)$ is the Riemann zeta function.

$$\zeta(n) = \sum_{p=1}^{\infty} p^{-n} = 1 + 2^{-n} + 3^{-n} + \cdots$$

$$\zeta(n) - 1 = 2^{-n} + 3^{-n} + \cdots = \sum_{p=2}^{\infty} p^{-n}$$

$$\sum_{n=2}^{\infty} [\zeta(n) - 1] = \sum_{n=2}^{\infty} \sum_{p=2}^{\infty} p^{-n} = \sum_{p=2}^{\infty} \underbrace{\left(\sum_{n=2}^{\infty} p^{-n} \right)}_{\text{geometric series}}$$

Define $A = \frac{1}{p^2} + \frac{1}{p^3} + \cdots$

then $\frac{1}{p} A = \frac{1}{p^3} + \frac{1}{p^4} + \cdots = A - \frac{1}{p^2}$

$$\Rightarrow A = \frac{1/p^2}{1 - 1/p} = \frac{p/p^2}{p-1} = \frac{1}{p(p-1)}$$

$$\sum_{n=2}^{\infty} [\zeta(n) - 1] = \sum_{p=2}^{\infty} \frac{1}{p(p-1)} = \sum_{p=2}^{\infty} \left[\frac{1}{p-1} - \frac{1}{p} \right] \rightarrow \text{telescoping series}$$

$$= \left[1 - \cancel{\tfrac{1}{2}}\right] + \left[\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right] + \left[\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right] + \cdots$$

$$= 1$$

(A-W 5.6.15)

The displacement x of a particle of rest mass m_0, resulting from a constant force $m_0 g$ along the x axis, is

$$x = \frac{c^2}{g}\left\{\left[1+\left(g\frac{t}{c}\right)^2\right]^{1/2}-1\right\}$$

including relativistic effects. Find the displacement x as a power series in time t. Compare with the classical result, $x=\frac{1}{2}gt^2$.

$$(1+y^2)^{1/2} = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}(y^2)^n$$ (binomial theorem, Taylor series)

where $m=\frac{1}{2}$

& $$\frac{(\frac{1}{2})!}{(\frac{1}{2}-n)!} = (\tfrac{1}{2})(\tfrac{1}{2}-1)\cdots(\tfrac{1}{2}-n+1)$$

→ you can think of this as notation, or in terms of Gamma functions

$$(1+y^2)^{1/2}-1 = \sum_{n=1}^{\infty}\frac{m!}{n!(m-n)!}(y^2)^n = \frac{1}{2}y^2 - \frac{1}{2}\frac{1}{4}(y^2)^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}(y^2)^3 + \cdots$$

so

$$x = \frac{c^2}{g}\left[\frac{1}{2}\left(\frac{gt}{c}\right)^2 - \frac{1}{2}\frac{1}{4}\left(\frac{gt}{c}\right)^4 + \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{gt}{c}\right)^6 + \cdots\right]$$

If we truncate to the first term, then

$$x \approx \frac{\cancel{c^2}}{\cancel{g}}\left(\frac{1}{2}\right)\frac{g^{\cancel{2}}t^2}{\cancel{c^2}} = \frac{1}{2}gt^2,$$ the classical result.

(A-W 6.2.1)

The functions $u(x,y)$, $v(x,y)$ are the real, imaginary parts of an analytic function $w(z)$.

a) Assuming the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's eq'n such as $u(x,y)$, $v(x,y)$ are called harmonic functions.

We know $u_x = v_y$, $u_y = -v_x$

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \Rightarrow \underline{u_{xx} + u_{yy} = 0, \ \nabla^2 u = 0}$$

$$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy} \Rightarrow \underline{v_{xx} + v_{yy} = 0, \ \nabla^2 v = 0}$$

b) Show that

$$u_x u_y + v_x v_y = 0$$

~~and then give a geometric interpretation~~

Cauchy-Riemann: $u_x = v_y, \quad u_y = -v_x$

$\Rightarrow u_x u_y = (v_y)(-v_x)$

$\Rightarrow u_x u_y + v_x v_y = 0$

(A-W) 6.2.3)

Having shown that the real part $u(x,y)$ & the imaginary part $v(x,y)$ of an analytic function $w(z)$ each satisfy Laplace's equ'n, show that $u(x,y)$ and $v(x,y)$ cannot both have either a maximum or a minimum in the interior of any region in which $w(z)$ is analytic.

S'pose at some point z_0,

$$u(z_0)_x = u(z_0)_y = v(z_0)_x = v(z_0)_y = 0$$

Define $D(u) = \det \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}$, $D(v) = \det \begin{bmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{bmatrix}$

$D > 0 \Rightarrow$ local maximum or minimum
$D < 0 \Rightarrow$ saddle point
$D = 0 \Rightarrow$ inconclusive

$$D(u) = u_{xx} u_{yy} - u_{xy} u_{yx}$$

but thanks to Cauchy-Riemann, $u_{xx} = -u_{yy}$, $u_{xy} = v_{yy}$

$$\Rightarrow D(u) = -(u_{xx})^2 - (v_{yy})^2$$

Similarly, $D(v) = -(v_{xx})^2 - (u_{yy})^2$

Both $D \leq 0$, so, no maxima or minima

Show that $e^{iz} = \cos z + i \sin z$ for every complex number z.

$$e^{iz} = 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

$$= \left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right] + i\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= \cos z + i \sin z$$

For $z = x + iy$, show that $|\sin z| \geq |\sin x|$.

In class, we showed

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$\geq \sin^2 x = |\sin x|^2$$

Result follows.

Find all roots of the equation $\cos z = 2$.

Recall from class that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

so $\cos z = 2 \Rightarrow$ $\cos x \cosh y = 2$
$\sin x \sinh y = 0$

$\sin x \sinh y = 0 \Rightarrow y = 0$ or $\sin x = 0$
But if $y=0$, then $\cos x \cosh y = 2$ has no sol'ns.

$\Rightarrow \sin x = 0 \Rightarrow x = n\pi$ for n an integer.

$\cos(n\pi) = (-)^n$
so $\cosh y = 2(-)^n$

If n is odd, no solutions.

$\Rightarrow$ Solutions of $\cos z = 2$ are

$$\boxed{z = 2n\pi + i \cosh^{-1}(2)}$$

Can push a bit further: compute $\cosh^{-1}(2)$.

Find y s.t. $\frac{1}{2}(e^y + e^{-y}) = 2 \Rightarrow e^y + e^{-y} = 4$
$\Rightarrow e^{2y} + 1 = 4e^y$

$\Rightarrow e^y = \frac{1}{2}(4 \pm \sqrt{16-4})$
$= 2 \pm \sqrt{3}$

so $y = \ln(2 \pm \sqrt{3})$

Note $\frac{1}{2-\sqrt{3}} \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3} \Rightarrow \ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$

so $y = \pm \ln(2+\sqrt{3})$

$\Rightarrow \boxed{z = 2n\pi \pm i \ln(2+\sqrt{3})}$

For a complex number z, define

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}), \qquad \cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

Show that $\sinh 2z = 2 \sinh z \cosh z$.

$$2 \sinh z \cosh z = 2\left(\frac{1}{2}\right)(e^{z} - e^{-z})\left(\frac{1}{2}\right)(e^{z} + e^{-z})$$

$$= \left(\frac{1}{2}\right)(e^{2z} - e^{-2z})$$

$$= \sinh 2z$$

For a complex number z,
Show that

$$-i\sinh(iz) = \sin z, \quad \cosh(iz) = \cos z$$

$$\sinh(iz) = \frac{1}{2}(e^{iz} - e^{-iz})$$

$$= i\left(\frac{1}{2i}\right)(e^{iz} - e^{-iz}) = i\sin z$$

$$\cosh(iz) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$= \cos z$$

For $z = x+iy$,
show that

$$\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$$= \tfrac{1}{2}(e^{z_1}-e^{-z_1})(\tfrac{1}{2})(e^{z_2}+e^{-z_2}) + (\tfrac{1}{2})(e^{z_1}+e^{-z_1})(\tfrac{1}{2})(e^{z_2}-e^{-z_2})$$

$$= (\tfrac{1}{4})\Big(e^{z_1+z_2} - e^{-(z_1+z_2)} - \cancel{e^{z_2-z_1}} + \cancel{e^{z_1-z_2}}$$

$$+ e^{z_1+z_2} - e^{-(z_1+z_2)} + \cancel{e^{z_2-z_1}} - \cancel{e^{z_1-z_2}}\Big)$$

$$= \tfrac{1}{2}\left(e^{z_1+z_2} - e^{-(z_1+z_2)}\right) = \underline{\sinh(z_1+z_2)}. \checkmark$$

$\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$$= (\tfrac{1}{4})(e^{z_1}+e^{-z_1})(e^{z_2}+e^{-z_2}) + (\tfrac{1}{4})(e^{z_1}-e^{-z_1})(e^{z_2}-e^{-z_2})$$

$$= (\tfrac{1}{4})\Big[e^{z_1+z_2} + e^{-(z_1+z_2)} + \cancel{e^{z_2-z_1}} + \cancel{e^{z_1-z_2}}$$

$$+ e^{z_1+z_2} + e^{-(z_1+z_2)} - \cancel{e^{z_2-z_1}} - \cancel{e^{z_1-z_2}}\Big]$$

$$= \tfrac{1}{2}\left[e^{z_1+z_2} + e^{-(z_1+z_2)}\right] = \underline{\cosh(z_1+z_2)} \checkmark$$

For $z = x + iy$,
show that

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$
$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

Recall $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

Take $z_1 = x$, $z_2 = iy$

$$\Rightarrow \sinh z = \sinh x \cosh(iy) + \cosh x \sinh(iy)$$
$$= \sinh x \cos y + i \cosh x \sin y$$ ✓

Recall $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

Take $z_1 = x$, $z_2 = iy$

$$\Rightarrow \cosh z = \cosh x \cosh(iy) + \sinh x \sinh(iy)$$

$$= \cosh x \cos y + i \sinh x \sin y$$ ✓

For $z = x + iy$,
show that

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

Recall $\sinh z = \sinh x \cos y + i \cosh x \sin y$

$$|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$

$$\cancel{= \sinh^2 x \,(1 - \sin^2 y)}$$

$$= \sinh^2 x \cos^2 y + (1 + \sinh^2 x) \sin^2 y$$

$$= \sinh^2 x (\cos^2 y + \sin^2 y) + \sin^2 y$$

$$= \underline{\sinh^2 x + \sin^2 y}$$

Recall $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$$|\cosh z|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y$$

$$= (1 + \sinh^2 x) \cos^2 y + \sinh^2 x \sin^2 y$$

$$= \sinh^2 x (\cos^2 y + \sin^2 y) + \cos^2 y$$

$$= \underline{\sinh^2 x + \cos^2 y}$$

7 —

1. Show that the holomorphic function

$$f_2(z) = \frac{1}{z^2+1} \qquad (z \neq \pm i)$$

is the analytic continuation of the function

$$f_1(z) = \sum_{n=0}^{\infty} (-)^n z^{2n} \qquad (|z| < 1)$$

into the domain <u>consisting of all points in the z plane except</u> $z = \pm i$.

f_1 is a geometric series, so sum it.

Define $A_N = \sum_{n=0}^{N} (-)^n z^{2n} = 1 + (-z^2) + (-z^2)^2 + \cdots + (-z^2)^N$

$$(-z^2) A_N = A_N - 1 + (-z^2)^{N+1}$$

$$\Rightarrow A_N = \frac{1 - (-z^2)^{N+1}}{1+z^2}$$

$$\Rightarrow f_1(z) = \lim_{N \to \infty} A_N = \frac{1}{1+z^2}$$ for $|z|<1$ where the geometric series converges.

Since $f_1 = f_2$ on $\{|z|<1\}$, & f_2 is defined outside of $\{|z|<1\}$, we see that f_2 is the analytic continuation of f_1.

2. Show that the function $f_2(z) = z^{-2}$ $(z \neq 0)$ is the analytic continuation of the function

$$f_1(z) = \sum_{n=0}^{\infty} (n+1)(z+1)^n \qquad (|z+1| < 1)$$

into the domain consisting of all points in the z plane except $z=0$.

Begin with the geometric series

$$A(z) = \sum_{n=0}^{\infty} (z+1)^{n+1} = (z+1) + (z+1)^2 + \cdots$$

Sum it:

Define $A_N = \sum_{n=0}^{N} (z+1)^{n+1} = (z+1) + \cdots + (z+1)^{N+1}$

$$(z+1) A_N = A_N - (z+1) + (z+1)^{N+2}$$

$$\Rightarrow A_N = \frac{(z+1) - (z+1)^{N+2}}{1-(z+1)}$$

$$\Rightarrow A(z) = \lim_{N \to \infty} A_N = \frac{z+1}{-z} = -1 - \frac{1}{z} \quad \text{for } |z+1|<1$$

Since $A(z)$ converges for $|z+1|<1$, on that same domain we can differentiate term-by-term:

$$\sum_{n=0}^{\infty} (n+1)(z+1)^n = \frac{\partial}{\partial z} A(z) = +\frac{1}{z^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)(z+1)^n = \frac{1}{z^2} \quad \text{on } |z+1|<1$$

Since $f_1 = f_2$ on $\{|z+1|<1\}$ & f_2 is defined outside of that region, f_2 is the analytic continuation of f_1.

3. Find the analytic continuation of the function

$$f(z) = \int_0^\infty t\, e^{-zt}\, dt \qquad (\operatorname{Re} z > 0)$$

into the domain consisting of all points in the z plane except the origin.

First, note $\int_0^\infty e^{-zt}\, dt = \frac{1}{z}$ for $\operatorname{Re} z > 0$.

In the region of convergence,

$$\frac{\partial}{\partial z}\int_0^\infty e^{-zt}\, dt = \int_0^\infty \left(\frac{\partial}{\partial z} e^{-zt}\right) dt = \int_0^\infty (-t)\, e^{-zt}\, dt$$

$$\underline{\text{also}} = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

$$\Rightarrow \int_0^\infty t\, e^{-zt}\, dt = \frac{1}{z^2} \quad \text{on } \operatorname{Re}(z) > 0$$

Since $\frac{1}{z^2}$ is well-defined for all nonzero z,

& it matches $\int_0^\infty t e^{-zt}\, dt$ on $\operatorname{Re}(z) > 0$,

we see $\frac{1}{z^2}$ is the analytic continuation.

4. Show that the function $(z^2+1)^{-1}$ is the analytic continuation of the function

$$f(z) = \int_0^\infty e^{-zt} \sin t \, dt \qquad (\operatorname{Re} z > 0)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

$$\int_0^\infty e^{-zt} \sin t \, dt = \int_0^\infty e^{-zt} \left[\frac{e^{it} - e^{-it}}{2i} \right] dt$$

$$= \frac{1}{2i} \int_0^\infty \left[e^{t(-z+i)} - e^{t(-z-i)} \right] dt$$

$$= \frac{1}{2i} \left[\frac{1}{-z+i}(-) - \frac{1}{-z-i}(-) \right] = -\frac{1}{2i} \left[\frac{1}{-z+i} + \frac{1}{z+i} \right]$$

$$= -\frac{1}{2i} \left[\frac{(\cancel{z}+i) + (-\cancel{z}+i)}{(-z+i)(z+i)} \right] = -\frac{1}{2i} \left(\frac{2i}{-1-z^2} \right)$$

$$= \frac{1}{1+z^2}$$

Thus, $f(z) = (1+z^2)^{-1}$ on $\{\operatorname{Re} z > 0\}$,

so we see that $(1+z^2)^{-1}$ is the analytic continuation of $f(z)$ to most of the rest of the complex plane.

1. Rodrigues' formula for the Legendre polynomials $P_n(z)$ says that

(20 pts)

$$P_n(z) = \frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n (z^2-1)^n \qquad n = 0, 1, 2, \ldots$$

a) Show that the Legendre polynomials can also be expressed as

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_c \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds \qquad n=0,1,2,\ldots$$

This is known as the Schlaefli integral.

First, recall $f^{(n)}(z) = \frac{n!}{2\pi i}\oint \frac{f(s)\,ds}{(s-z)^{n+1}}$ (AW (6.47))

$$\Rightarrow \frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n (z^2-1)^n = \frac{1}{2^n n!}\,\frac{n!}{2\pi i}\oint \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds$$

$$= \frac{1}{2^{n+1}\pi i}\oint \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds$$

b) Show that $P_n(1) = 1$ and $P_n(-1) = (-)^n$ for all n, using the Schlaefli integral representation of $P_n(z)$.

First, note $(s^2-1)^n = (s+1)^n(s-1)^n$

So,

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\oint \frac{(s+1)^n(s-1)^n}{(s-1)^{n+1}}\,ds = \frac{1}{2^{n+1}\pi i}\oint \frac{(s+1)^n}{s-1}\,ds$$

Recall $f(z) = \frac{1}{2\pi i}\oint \frac{f(s)}{s-z}\,ds$

$$\Rightarrow P_n(1) = \frac{1}{2^n}\left[(s+1)^n\Big|_{s=1}\right] = \frac{1}{2^n}\,2^n = \underline{1}$$ ✓

Similarly,

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\oint \frac{(s+1)^n(s-1)^n}{(s+1)^{n+1}}\,ds = \frac{1}{2^{n+1}\pi i}\oint \frac{(s-1)^n}{s+1}\,ds$$

$$= \frac{1}{2^n}\left[(s-1)^n\Big|_{s=-1}\right] = \frac{1}{2^n}(-2)^n = \underline{(-)^n}$$ ✓

Riemann surfaces

2\. Describe a Riemann surface for the triple-valued function

10 pts

$$w = (z-1)^{1/3}$$

and point out which ~~sheet~~ third of the w plane represents the image of each sheet of the surface.

Take 3 copies of the complex plane, label them A, B, C.
Cut each along the real axis from $z=1$ to $+\infty$.

The A sheet ~~should~~ has $0 \le \arg z < 2\pi$
The B " " $2\pi \le \arg z < 4\pi$
The C " " $4\pi \le \arg z < 6\pi$

so, glue the lower edge of the cut on the A sheet to upper edge on B sheet
" " " " " " " " " B " " " " " C "
" " " " " " " " " C " " " " " A "

On the w plane, $0 \le \arg w < \frac{2\pi}{3}$ is the image of ~~corresponds to~~ A sheet
$\frac{2\pi}{3} \le$ " $< \frac{4\pi}{3}$ " " " " B sheet
$\frac{4\pi}{3} \le$ " $< 2\pi$ " " " " C sheet

Riemann surfaces

3. Describe the curve, on a Riemann surface for $z^{1/2}$,
10 pts
whose image is the entire circle $|w| = 1$ under the transformation
$w = z^{1/2}$.

The curve ~~follows~~ lies on $|z| = 1$.
It starts on the first sheet at $\arg z = 0$,
revolves 2π around origin,
goes onto the second sheet, and revolves 2π around the
origin before returning to its starting position.

If the curve only lied on one sheet,
then its image on the w plane would only be a semicircle,
not a full circle.