Physics 5714 – Problem set 4

1. Find three vectors in $\mathbf{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

2. Let V be the vector space of all 2×2 matrices over a field F. Show that V has dimension 4 by exhibiting a basis with 4 elements.

3. Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, γ are linearly independent vectors in V. Show that $(\alpha + \beta)$, $(\beta + \gamma)$, $(\gamma + \alpha)$ are linearly independent.

4. Show that the vectors

$$\begin{aligned} \alpha_1 &= (1, 1, 0, 0) \\ \alpha_2 &= (0, 0, 1, 1) \\ \alpha_3 &= (1, 0, 0, 4) \\ \alpha_4 &= (0, 0, 0, 2) \end{aligned}$$

form a basis for $\mathbf{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

5. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbf{R}^3$ consisting of

$$\begin{aligned} \alpha_1 &= (1, 0, -1) \\ \alpha_2 &= (1, 1, 1) \\ \alpha_3 &= (1, 0, 0) \end{aligned}$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

6. Let V be the vector space over the complex numbers of all functions $\mathbf{R} \to \mathbf{C}$, *i.e.* the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = \exp(ix)$, $f_3(x) = \exp(-ix)$.

   (a) Show that f_1, f_2, f_3 are linearly independent.

   (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j(x) = \sum_i P_{ij} f_i(x)$$

7. Let V be the real vector space of all polynomial functions $\mathbf{R} \to \mathbf{R}$ of degree 2 or less, *i.e.* the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Let t be a fixed real number, and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x + t)^2$$

Show that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of f in this ordered basis $\mathcal{B}$?

8. Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$, $\alpha_3 = (-1, 2, 5, 2)$. Let $\alpha = (4, -5, 9, -7)$, $\beta = (3, 1, -4, 4)$, $\gamma = (-1, 1, 0, 1)$. Which of the vectors α, β, γ are in the subspace of $\mathbf{R}^4$ spanned by the α_i ?

9. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional vector space V.

10. Describe the range and null space of the differentiation transformation on the vector space of polynomials.

11. Is there a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, -1, 1) = (1, 0)$, $T(1, 1, 1) = (0, 1)$?

12. Let V be the vector space of all $n \times n$ matrices over a field F, and let B be a fixed $n \times n$ matrix. If
$$T(A) = AB - BA$$
verify that T is a linear transformation $V \to V$.

13. Let V be the set of all complex numbers regarded as a vector space over the field of real numbers. Find a function $V \to V$ which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbf{C}$, *i.e.* which is not complex linear.

14. Let V be an n-dimensional vector space over the field F, T a linear transformation $V \to V$ such that range T = null space T. Show that n is even.

15. Let V be a vector space, $T : V \to V$ a linear transformation. Show that the following statements are equivalent:

    (a) the intersection of the range of T and the null space of T is the zero subspace of V

    (b) if $T(T\alpha) = 0$ then $T\alpha = 0$