Matrix Theory

(§3.1 Linear transformations)

<u>Def'n</u> Let V, W be vector spaces over the field F.
A <u>linear transformation</u> from V into W
is a function $T: V \to W$ such that
$$T(c\alpha + \beta) = c(T\alpha) + T\beta \qquad \forall \alpha, \beta \in V, \ \forall c \in F$$

<u>Ex</u> For any vector space V,
the <u>identity transformation</u> $I: \alpha \mapsto \alpha$
the zero transformation $0: \alpha \mapsto 0$

– check both are linear

<u>Ex</u> Let V be the space of polynomials over field F,
commonly denoted $F[x]$.

The differentiation transformation D takes derivatives:
for $f(x) = c_0 + c_1 x + \cdots + c_n x^n$,
$$(Df)(x) = c_1 + 2c_2 x + \cdots + n c_n x^{n-1}$$

– check linear

<u>Ex</u> Let A be an $m \times n$ matrix,
$T: F^{n\times 1} \to F^{m \times 1}$ by $T\alpha = A\alpha$

– check linear.

<u>Ex</u> Let V = vector space of cont' functions $\mathbb{R} \to \mathbb{R}$.
Define $(Tf)(x) = \int_0^x f(t)\,dt$

– check linear

Note $T(0)=0$:

$$T(0) = T(0+0) = T(0) + T(0)$$

Also $T(a-a) = T(0)$
$= T(a) - T(a)$
$= 0$

So $T: x \mapsto ax+b$ for $b \neq 0$

is not linear!

$\rightsquigarrow$ so watch out, this notion of linear maybe slightly counterintuitive

Also note linear trans' preserve linear combinations:

$$T(c_1\alpha_1 + \cdots + c_n\alpha_n) = c_1(T\alpha_1) + \cdots + c_n(T\alpha_n)$$

Thm Let V be a finite-dim'l vector space,
let $\{\alpha_1, \dots \alpha_n\}$ be an ordered basis for V,
Let W be a vector space over the same field,
& let $\beta_1, \dots \beta_n$ be any vectors in W,

Then there is precisely one linear transformation $T: V \to W$
st $T\alpha_i = \beta_i \quad \forall i$

Since $\{\alpha_1, \dots \alpha_n\}$ form a basis,
any vector $\alpha \in V$ can be written $\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$.

Define $T\alpha = c_1\beta_1 + \dots + c_n\beta_n$

$\rightsquigarrow$ check linear.

Furthermore, if $U: V \to W$ is any other lin' trans' st $U\alpha_i = \beta_i$,
then, for $\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$,

$U\alpha = c_1\beta_1 + \dots + c_n\beta_n$ by linearity
$= T\alpha$

so $U = T$,
& the lin' trans' is unique.

Ex The vectors $\alpha_1 = (1,2)$, $\alpha_2 = (8,7)$ form a basis for $\mathbb{R}^2$.

(Check: why LI? $\rightsquigarrow$ b/c not proportional)

$\exists!$ lin' trans $\mathbb{R}^2 \to \mathbb{R}^3$ s.t. $T\alpha_1 = (1,2,3)$
$T\alpha_2 = (0,0,1)$

What is $T(1,0)$?

1^{st}, write $(1,0) = c_1(1,2) + c_2(8,7)$

$$\Rightarrow c_1 + 8c_2 = 1, \quad 2c_1 + 7c_2 = 0$$
$$\Rightarrow c_2 = -\frac{2}{7}c_1$$
$$\Rightarrow c_1 - \frac{16}{7}c_1 = 1 \Rightarrow c_1 = -\frac{7}{9}, \quad c_2 = +\frac{2}{9}$$

so $T(1,0) = -\frac{7}{9}(1,2,3) + \frac{2}{9}(0,0,1)$

Ex Recall T !'ly determined by images of a basis
— so take standard basis.

Define $\beta_i = T\varepsilon_i$, $\quad \varepsilon_i = (0, \cdots 1, \cdots 0)$
$\hookrightarrow i^{th}$ position

Then T can be represented by a matrix:

$$T(c_1\varepsilon_1 + \cdots + c_n\varepsilon_n) = c_1\beta_1 + \cdots + c_n\beta_n$$

becomes

$$T\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_1 \,|\, \beta_2 \,|\, \cdots \,|\, \beta_n \end{bmatrix}}_{\sim T}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\rightsquigarrow$ will come back to this later.

If $T: V \to W$ is a linear transformation,
then, $\text{im}\, T$ is a subspace of W: (called range)

suppose $\alpha, \beta \in \text{im}\, T$, so $\exists$ $a, b \in V$ st $\alpha = Ta$, $\beta = Tb$.
then $c\alpha + \beta = cTa + Tb = T(ca+b) \in \text{im}\, T$
& nonempty b/c $T(0) = 0$.

The null space of a linear trans' $T: V \to W$
is the set of vectors $\alpha \in V$ s.t. $T\alpha = 0$.

Claim the null space is a subspace of V:

- nonempty since $T(0) = 0$
- if $\alpha, \beta \in$ null space,
  then $c\alpha + \beta \in$ null space since $T(c\alpha + \beta) = 0$.

Define rank of a linear tran' T = dim range
nullity " " " " = dim null space.

Thm Let V, W be vector spaces over field F
and let $T: V \to W$ be a linear transformation.
S'pose V is finite-dim'l.
Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

Pf Let $\{\alpha_1, \dots \alpha_k\}$ be a basis for the null space of T.
There are vectors $\alpha_{k+1}, \dots \alpha_n$ st $\{\alpha_1, \dots \alpha_n\}$ is a basis for V.
Claim $\{T\alpha_{k+1}, \dots T\alpha_n\}$ is a basis for the range of T.

Certainly $T\alpha_1, \dots T\alpha_n$ span the range of T,
& since $T\alpha_j = 0$ for $j \le k$, $T\alpha_{k+1}, \dots T\alpha_n$ span the range of T.
Now, show LI.

S'pose $c_{k+1} T\alpha_{k+1} + \dots + c_n T\alpha_n = 0$

$$\Rightarrow T(c_{k+1}\alpha_{k+1} + \dots + c_n \alpha_n) = 0$$

$$\Rightarrow c_{k+1}\alpha_{k+1} + \dots + c_n\alpha_n \in \text{null space of } T.$$

Since $\alpha_1, \dots \alpha_k$ form a basis for the null space,
$\exists\ b_i$ s.t.

$$c_{k+1}\alpha_{k+1} + \dots + c_n\alpha_n = b_1\alpha_1 + \dots + b_k\alpha_k$$

$$\Rightarrow b_1\alpha_1 + \dots + b_k\alpha_k - c_{k+1}\alpha_{k+1} - \dots - c_n\alpha_n = 0$$

But the α_i are LI $\Rightarrow \underline{b_i = c_j = 0}$

$\Rightarrow T\alpha_{k+1}, \dots T\alpha_n$ are LI, hence form a basis.

~~[illegible]~~

~~[illegible]~~

Start here Mon

Thm Let A be an $m \times n$ matrix.
Then: row rank (A) = column rank (A).

Pf

Let T be the linear transformation $F^{n \times 1} \to F^{m \times 1}$ defined by $T(X) = AX$.

Null space of $T = \{X \mid AX = 0\}$.

If $A_1, \dots A_n$ are the columns of A, then $AX = x_1 A_1 + \dots + x_n A_n$.

$\Rightarrow$ range T = column space of A

$\Rightarrow$ rank T = column rank A

(see p. 42 then → 72)

~~Now, consider putting A in row-echelon form.~~

~~row rank (A) + (# zero rows) =~~

Let S be the solution space of the system $\{AX = 0\}$. (= null space)

~~row rank = #~~

Put A in row reduced echelon form, call it R.

(Same sol'n space S.)

row rank = # nontrivial eqn's (others identically zero)

n = # unknowns

so; $\dim S = n -$ row rank (A)

Also, since S = null space of T,

$$\underbrace{\dim S}_{= \text{nullity } T} + \text{rank } T = n$$

so $\dim S = n - \text{rank } T = n -$ row rank A

Since rank T = column rank A,
we see

column rank A = row rank A

(§3.2 Algebra of linear transformations)

<u>Thm</u> Let V, W be vector spaces over a field.
Let T, U be linear transformations $V \to W$.

- The function $(T+U)$ defined by
$$(T+U)(\alpha) = \cancel{T\alpha + U\alpha} = T\alpha + U\alpha$$
is a linear transformation $V \to W$.
- If c is a scalar, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation.
- The set of all linear transformations $V \to W$ is a vector space.

<u>Pf</u>

- Check $(T+U)(c\alpha+\beta) = c(T+U)(\alpha) + (T+U)(\beta)$
- Ditto for cT

- Vector space: mostly clear (outline some on board).
Zero vector = zero transformation $0 : \alpha \mapsto 0$

<u>Notation</u> $L(V, W)$ = vector space of all linear trans' $V \to W$.

Thm Let V be an n-dim'l vector space over F,
& W an m-dim'l " " " "
Then the space $L(V,W)$ is finite-dim'l & of dimension mn.

Pf
Let $\mathcal{B} = \{\alpha_1, \cdots \alpha_n\}$, $\mathcal{B}' = \{\beta_1, \cdots \beta_m\}$
be ordered bases for V, W, resp.

For each pair of integers (p,q), $1 \le p \le m$, $1 \le q \le n$,
define a linear transformation $E^{p,q}: V \to W$ by

$$E^{p,q}(\alpha_i) = \delta_{iq} \beta_p$$

(There is ! lin' trans' w/ this property.)

Claim the $E^{p,q}$ form a basis for $L(V,W)$.

Check span:
Let $T: V \to W$ be a linear transformation.
For each j, $1 \le j \le n$, let $A_{1j}, \cdots A_{mj}$ be the coord's of $T\alpha_j$ in basis $\mathcal{B}'$,
i.e., $T\alpha_j = \sum_p A_{pj} \beta_p$

Claim $T = \sum_p \sum_q A_{pq} E^{p,q}$

Check: $\sum_p \sum_q A_{pq} E^{p,q}(\alpha_j) = \sum_{p,q} A_{pq} \delta_{jq} \beta_p = \sum_p A_{pj} \beta_p = T\alpha_j$

& so $T = \sum_{p,q} A_{pq} E^{p,q}$

$\Rightarrow E^{p,q}$ span $L(V,W)$

Need to show LI:
Suppose $\sum_{p,q} A_{pq} E^{p,q} = 0$ transformation
$\Rightarrow \sum_{p,q} A_{pq} E^{p,q}(\alpha_j) = 0 \ \forall j \Rightarrow \sum_p A_{pj} \beta_p = 0$

Since β's are LI, $\Rightarrow A_{pj} = 0 \ \forall p, j$.

Thm Let V, W, Z be vector spaces over a field F.
Let T be a linear transformation $V \to W$ $U: W \to Z$ a lin' trans'.
Define the composition $(UT)(\alpha) = U(T(\alpha))$.
Then UT is a linear trans' $V \to Z$.

– check on board.

Def'n A lin' trans' $V \to V$ is called a linear operator on V.

Lemma Let V be a vector space over a field F,
let U, T_1, T_2 be linear operators on V, let $c \in F$.

- $IU = UI = U$
- $U(T_1 + T_2) = UT_1 + UT_2$
  $(T_1 + T_2)U = T_1 U + T_2 U$
- $c(UT_1) = (cU)T_1 = U(cT_1)$

Check parts of this:

$$\begin{aligned} U(T_1 + T_2)(\alpha) &= U(T_1\alpha + T_2\alpha) && \text{by def'n of } + \\ &= U(T_1\alpha) + U(T_2\alpha) && \text{by linearity of } U \\ &= (UT_1)(\alpha) + (UT_2)(\alpha) && \text{by def'n of product.} \end{aligned}$$

& so forth.
