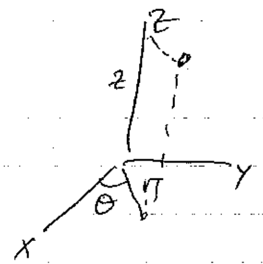
# § 14.7 Cylindrical & spherical coordinates

**Cylindrical coord's:** $(r, \theta, z)$




$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$
$$z = z \qquad z = z$$

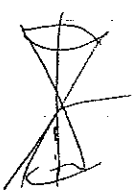
- convert some sample coords back & forth
- include $r<0$ cases

~~Ex Find eqn's in cylindrical coord's for paraboloid~~ ellipsoid ~~& cylinder whose Cartesian eqn's are~~

Ex Find eqn's in cylindrical coord's for ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \rightarrow \quad r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + \frac{z^2}{c^2} = 1$$

For elliptic cone



$$x^2+y^2 - \frac{z^2}{c^2} = 0 \quad \Rightarrow \quad r^2 - \frac{z^2}{c^2} = 0 \quad \text{or} \quad \underline{z = \pm cr}$$

(sketch)

Elliptic paraboloid

$$z = x^2+y^2 \quad \rightarrow \quad z = r^2 \qquad \text{(sketch)}$$

Ex Find Cartesian eqn's:

- $r^2 = z^2 + 1 \quad \to \quad x^2 + y^2 = z^2 + 1$
- $r^2 \sin 2\theta = z \quad \to \quad r^2(2 \sin\theta \cos\theta) = z$

  or $2xy = z$

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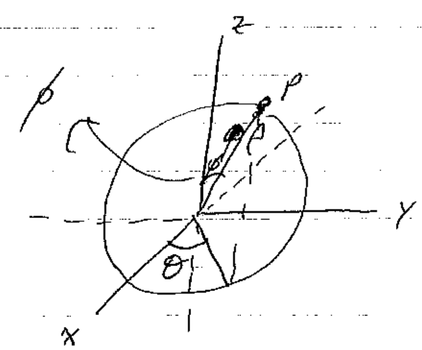
Start here Fri

Spherical coord's




$\theta$ = angle in xy plane

$\phi$ = angle off z axis

~~$(\rho, \phi, \theta)$~~ $(\rho, \theta, \phi)$

Usually take $0 \le \phi \le \pi$ (not $2\pi$)

$0 \le \theta < 2\pi$

(explain redundancy)

Explain:

$x = \rho \sin\phi \cos\theta \qquad \rho = \sqrt{x^2+y^2+z^2}$

$y = \rho \sin\phi \sin\theta \qquad \tan\theta = y/x$

$z = \rho \cos\phi \qquad \cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

– convert some sample coord's back & forth

Ex Convert eqn for a sphere: $x^2+y^2+z^2 = R^2 \quad \to \rho^2 = R^2$

Ex Convert $\rho = 5 \sin\phi$

$\to \rho^2 = 5\rho \sin\phi \quad \to \quad x^2+y^2+z^2 = 5\sqrt{x^2+y^2}$

Ex Convert $z = x^2+y^2$

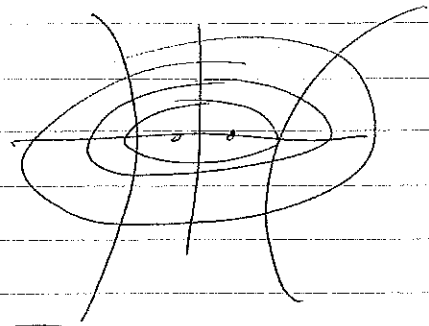
$\to \rho\cos\phi = \rho^2 \sin^2\phi \quad \to \quad \underline{\rho \sin^2\phi = \cos\phi}$

MFvI
pp 656-

Did this Wed

Culture

Elliptic coordinates: & elliptic cylindrical



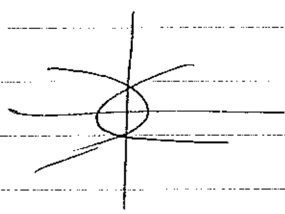
$X = c\cosh\xi\cos\eta,\quad Y = c\sinh\xi\sin\eta$

$\xi$ constant: $\dfrac{X^2}{c^2\cosh^2\xi} + \dfrac{Y^2}{c^2\sinh^2\xi} = 1 \rightarrow$ ellipse

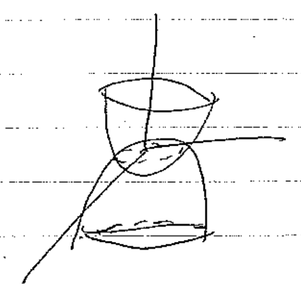
$\eta$ constant: $\dfrac{X^2}{c^2\cos^2\eta} - \dfrac{Y^2}{c^2\sin^2\eta} = 1 \rightarrow$ hyperbola

Parabolic: $0 \le \xi < \infty,\quad 0 \le \eta < 2\pi$

& parabolic cylindrical



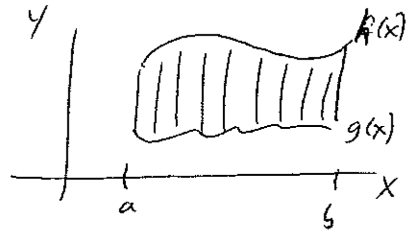
Parabolic in 3D:



Prolate spheroidal, oblate spheroidal, ellipsoidal, paraboloidal

## 16.3 Double integrals over nonrectangular regions

Consider integrating over region below:




Slice vertically, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

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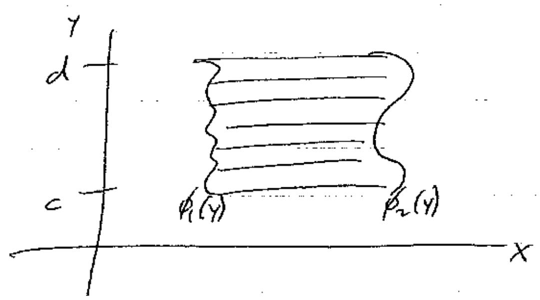
Integrate over region:




Slice horizontally.

$$\iint_R f(x,y)\,dA = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx\,dy$$

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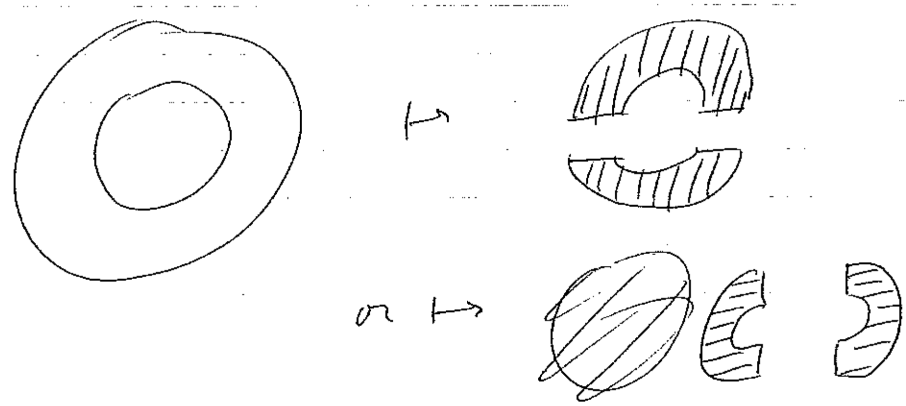
Break more complicated regions into regions of this form.

<u>Ex</u>



Ex Evaluate

$$\int_0^1 \int_x^{x^2} xy \, dy \, dx$$

$$= \int_0^1 \left[\frac{1}{2} x y^2\right]_{y=x}^{y=x^2} dx$$

$$= \int_0^1 \left[\frac{1}{2} x (x^4 - x^2)\right] dx = \int_0^1 \left(\frac{1}{2} x^5 - \frac{1}{2} x^3\right) dx$$

$$= \left[\frac{1}{2}\frac{1}{6} x^6 - \frac{1}{2}\frac{1}{4} x^4\right]_0^1$$

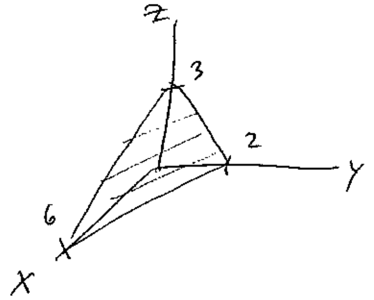
$$= \frac{1}{2}\frac{1}{6} - \frac{1}{2}\frac{1}{4}$$

Note $\int_x^{x^2} \int_0^1 xy \, dy \, dx$ makes no sense!

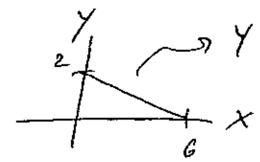
Order is critical!

Ex Find volume of tetrahedron bdd by coordinate planes & by the plane $x + 3y + 2z = 6$, in $1^{st}$ octant.

$1^{st}$, draw a picture.

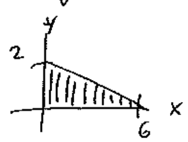



Volume lies over the triangle




$y = 2 - \frac{1}{3}x$

(A) Slice vertically.




$$\text{Volume} = \int_0^6 \int_0^{2-\frac{1}{3}x} \left[3 - \tfrac{1}{2}x - \tfrac{3}{2}y\right] dy\, dx$$

$$= \int_0^6 \left[3y - \tfrac{1}{2}xy - \tfrac{3}{4}y^2\right]_{y=0}^{y=2-\frac{1}{3}x} dx$$

$$= \int_0^6 \left[3(2 - \tfrac{1}{3}x) - \tfrac{1}{2}x(2 - \tfrac{1}{3}x) - \tfrac{3}{4}(2 - \tfrac{1}{3}x)^2\right] dx$$

$$= \int_0^6 \left[6 - x - x + \tfrac{1}{6}x^2 - \tfrac{3}{4}\left(4 - \tfrac{4}{3}x + \tfrac{1}{9}x^2\right)\right] dx$$

$$= \int_0^6 \left[3 - x + \tfrac{1}{12}x^2\right] dx$$

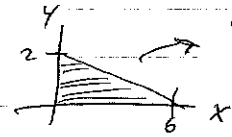
$$= \left[3x - \tfrac{1}{2}x^2 + \tfrac{1}{12}\tfrac{1}{3}x^3\right]_0^6$$

$$= 3(6) - \tfrac{1}{2}(6)^2 + \tfrac{1}{12}\tfrac{1}{3}(6)^3 = 18 - 18 + \tfrac{1}{6}\tfrac{1}{6}6^3 = \underline{6}$$

(cont'd)

Ex, cont'd

(B) Slice horizontally.




$y = 2 - \frac{1}{3}x$

or, $x = 6 - 3y$

Should get the same answer, regardless.

$$\text{Volume} = \int_0^2 \int_0^{6-3y} \left[3 - \tfrac{1}{2}x - \tfrac{3}{2}y\right] dx\, dy$$

$$= \int_0^2 \left[3x - \tfrac{1}{2}\tfrac{1}{2}x^2 - \tfrac{3}{2}yx\right]_{x=0}^{x=6-3y} dy$$

$$= \int_0^2 \left[3(6-3y) - \tfrac{1}{4}(6-3y)^2 - \tfrac{3}{2}y(6-3y)\right] dy$$

$$= \int_0^2 \left[18 - 9y - \tfrac{1}{4}(36 - 36y + 9y^2) - \tfrac{18}{2}y + \tfrac{9}{2}y^2\right] dy$$

$$= \int_0^2 \left[9 - 9y + \tfrac{9}{4}y^2\right] dy$$

$$= \left[9y - \tfrac{9}{2}y^2 + \tfrac{9}{4}\tfrac{1}{3}y^3\right]_0^2$$

$$= 9(2) - \tfrac{9}{2}(2)^2 + \tfrac{9}{4}\tfrac{1}{3}(2)^3 = 9 \cdot 2/3 = \underline{6}$$

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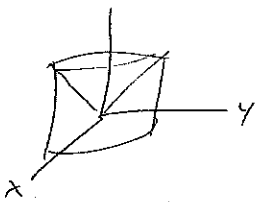
So we've found

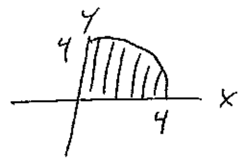
$$\int_0^6 \int_0^{2-\frac{1}{3}x} \left[3 - \tfrac{1}{2}x - \tfrac{3}{2}y\right] dy\, dx = \int_0^2 \int_0^{6-3y} \left[3 - \tfrac{1}{2}x - \tfrac{3}{2}y\right] dx\, dy$$

→ computing same volume,
but, if base not rectangular,
then changing order of integration <u>non</u> trivial.

Can <u>not</u> just exchange order: $\int_0^{2-\frac{1}{3}x} \int_0^6 [\;] dx\, dy$ doesn't even make sense!

Ex Find volume bdd by circular paraboloid $z = x^2+y^2$, cylinder $x^2+y^2=16$, & coord planes.




Slice vertically.

$$\int_0^4 \int_0^{\sqrt{16-x^2}} x^2+y^2 \, dy \, dx$$

$$= \int_0^4 \left[x^2 y + \tfrac{1}{3} y^3\right]_{y=0}^{y=\sqrt{16-x^2}} dx$$

$$= \int_0^4 \left[ x^2\sqrt{16-x^2} + \tfrac{1}{3}(16-x^2)^{3/2}\right] dx$$

Write $x = 4\sin\theta$

$dx = 4\cos\theta \, d\theta$

$$= \int_0^{\pi/2} \left[16\sin^2\theta\,(4)\cos\theta + \tfrac{1}{3}(4)^{3/2}\cos^3\theta\right](4\cos\theta)\,d\theta$$

$$\cancel{= \int_0^{\pi/2} [64\sin^2\theta\cos^2\theta}$$

$$= \int_0^{\pi/2} \left[(16)^2\sin^2\theta\cos^2\theta + \tfrac{1}{3}4^{5/2}\cos^4\theta\right] d\theta$$

Can finish w/ $\sin^2\theta\cos^2\theta = \left(\frac{1}{2}\right)^2 \sin^2 2\theta$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

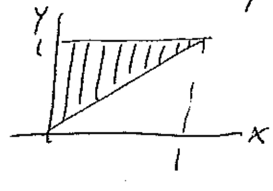
$\rightarrow$ don't finish on the board, leave the rest as an exercise.

Start here 1/Mon

Ex Compute

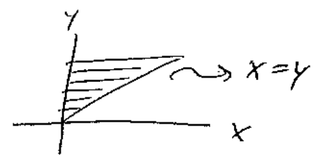
$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$$

Can't do directly – change order of integration.
The region described is




sliced vertically.

Slice horizontally instead:




$$= \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$= \int_0^1 \left[x e^{y^2}\right]_{x=0}^{x=y} dy \quad = \int_0^1 y e^{y^2} \quad = \tfrac{1}{2} e^{y^2}\Big|_0^1$$

$$= \tfrac{1}{2}(e^1 - 1) = \underline{\tfrac{1}{2}(e-1)}$$

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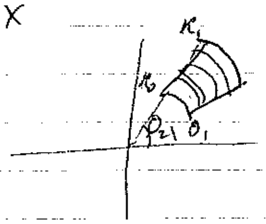
## 16.4 Double integrals in polar coordinates

$dA = (dr)(r\,d\theta)$ (explain intuitively)

Ex



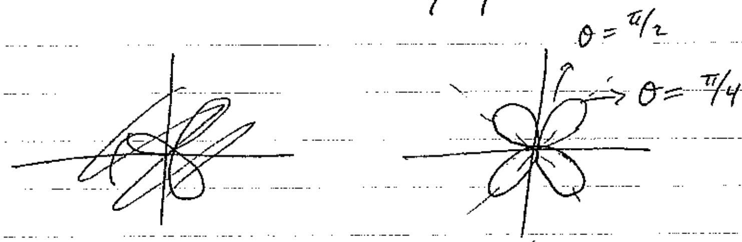

To integrate over that area:

$$\int_{R_0}^{R_1}\int_{\theta_1}^{\theta_2} f(x,y)\, r\,d\theta\,dr$$

→ analogue of a rectangle,
in that endpts are constant,
& can be triv'ly exchanged: $\int_{R_0}^{R_1}\int_{\theta_1}^{\theta_2} f\,r\,d\theta\,dr = \int_{\theta_1}^{\theta_2}\int_{R_0}^{R_1} f\,r\,dr\,d\theta$

Text calls such a pie wedge a <u>polar rectangle</u>.

<u>Ex</u> Find the area in one leaf of




Volume of cylinder = (area base)(height)

so Area $= \iint (1)\,dA$

$$= 4\int_0^{\pi/2}\int_0^{4\sin 2\theta} (1)\,r\,dr\,d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

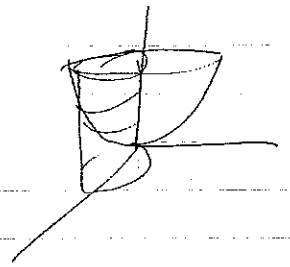
$$= 4\int_0^{\pi/2} \tfrac{1}{2}(4\sin 2\theta)^2\,d\theta = \tfrac{1}{2}(4)^3\int_0^{\pi/2}\tfrac{1}{2}[1-\cos 4\theta]\,d\theta$$

$$= \tfrac{1}{4}(4)^3\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= (4)^2(\pi/2)$$

$$= 8\pi$$

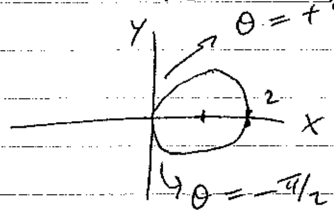
Ex Find the volume of the solid above $z = x^2 + y^2$, below $z = 5$, & inside the cylinder $x^2 + y^2 = 2x$.



$$x^2 + y^2 = 2x \iff r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$

$$z = x^2 + y^2 \iff z = r^2$$

$r = 2\cos\theta$:




$$\text{Volume} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (5 - r^2)\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[ \frac{5}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[ \frac{5}{2}(2\cos\theta)^2 - \frac{1}{4}(2\cos\theta)^4 \right] d\theta$$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$= \int_{-\pi/2}^{\pi/2} \left\{ \frac{5}{2}(4)\left(\frac{1}{2}\right)(1 + \cos 2\theta) - \frac{1}{4}(2)^4\left(\frac{1}{2}\right)^2(1 + \cos 2\theta)^2 \right\} d\theta$$

$$= \left[ 5\theta + 0 - \left(1 + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) \right]_{-\pi/2}^{\pi/2}$$

$$= 5(\pi) - \frac{3}{2}(\pi) = \underline{\frac{7}{2}\pi}$$

Ex Compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

[Recall improper integrals, then gloss over.]

Trick define $I$ = integral above

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

→ but this is a double integral over plane, so evaluate as such.

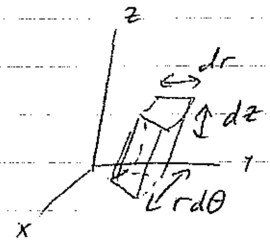
$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\right]d\theta$$

$$= \int_0^{2\pi}\left[+\frac{1}{2}\right]d\theta = \pi$$

$\Rightarrow$ $I^2 = \pi$ so $\underline{I = +\sqrt{\pi}}$ (not $-\sqrt{\pi}$ b/c positive)

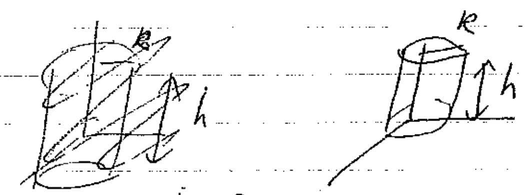
## 16.8 Triple integrals (Cylindrical & spherical coord's)

Cylindrical coord's




$$dV = \cancel{r\,dr\,d}\;(r\,d\theta)(dr)(dz) = r\,dr\,d\theta\,dz$$

Ex Compute volume of a cylinder

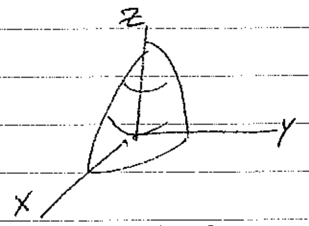


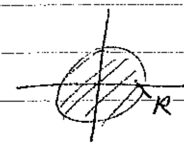

$$= \int_0^h \int_0^{2\pi} \int_0^R (1)\; r\,dr\,d\theta\,dz = \int_0^h \int_0^{2\pi} \frac{1}{2} R^2\, d\theta\, dz$$

$$= \frac{1}{2} R^2 (2\pi) \int_0^h dz = \pi R^2 h \quad \checkmark$$

$\longrightarrow$ exactly the standard geometric result.

Ex Compute volume inside paraboloid $z = R^2 - x^2 - y^2$
& above xy plane.



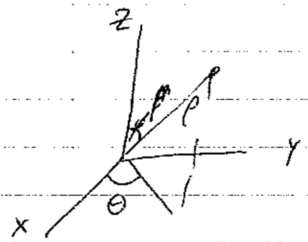

$$\text{Volume} = \int_0^{2\pi}\int_0^{R}\int_0^{R^2-r^2} (1)\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{R} (R^2 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=R} d\theta$$

$$= (2\pi)\left[\frac{1}{2}R^4 - \frac{1}{4}R^4\right] = \frac{2\pi}{4}R^4 = \underline{\frac{\pi}{2}R^4}$$

Spherical coordinates:



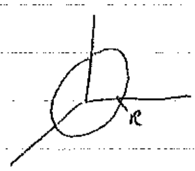

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

$$dV = \underbrace{(\rho \sin\phi \, d\theta)}_{\sim r}(\rho\, d\phi)(d\rho) = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

→ explain, "spherical wedge"

Ex. Volume of a sphere of radius R centered at origin.




$$\text{Volume} = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} (1)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

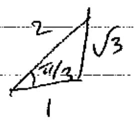
$$= \int_0^{\pi}\int_0^{2\pi} \tfrac{1}{3} R^3 \sin\phi \, d\theta \, d\phi$$

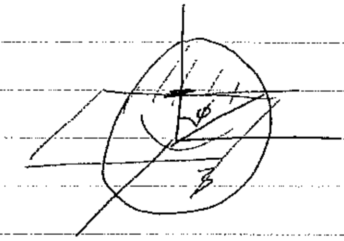
$$= \int_0^{\pi} \frac{2\pi}{3} R^3 \sin\phi \, d\phi$$

$$= \frac{2\pi}{3} R^3 \left(-\cos\phi\right)_0^{\pi} = \frac{2\pi}{3} R^3 (1 - -1)$$

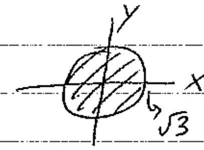
$$= \frac{4\pi}{3} R^3,$$ exactly the standard geometric result ✓

Ex Find the volume of that part of the sphere $x^2+y^2+z^2=4$ that lies above the plane $z=1$.




Lies above




Largest $\phi = \arccos(1/2)$
$= \pi/3$

$z = 1$
$= \rho\cos\phi$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\arccos(1/2)}\int_{\sec\phi}^{2} (1)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\arccos(1/2)} \frac{1}{3}\left[2^3 - \sec^3\phi\right]\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(\frac{1}{3}\right)(8)\sin\phi\, d\phi\, d\theta - \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \tan\phi\sec^2\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{8}{3}\right)\left(-\cos\phi\right)\Big]_{\phi=0}^{\phi=\pi/3} d\theta - \frac{1}{3}\int_0^{2\pi}\frac{1}{2}\left[\tan^2\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta$$

$$= (2\pi)(8/3)\left(-\frac{1}{2} - -1\right) - \frac{1}{3}\,\frac{1}{2}\,(2\pi)\left[(\sqrt{3})^2 - 0^2\right]$$

$$= (\cancel{2}\pi)(8/3)(1/\cancel{2}) - \frac{2\pi}{6}(3) = \frac{8\pi}{3} - \pi = \frac{\pi}{3}(8-3)$$

$$= \frac{5\pi}{3}$$