Analytic Continuation

See Arfken & Weber pp 432-434 (in section 6.5 on Laurent expansions) for some of the material below. Our description here will closely follow [1].

1 Definition

The *intersection* of two domains (regions in the complex plane) D_1, D_2, denoted $D_1 \cap D_2$, is the set of all points common to both D_1 and D_2. The *union* of two domains D_1, D_2, denoted $D_1 \cup D_2$, is the set of all points in either D_1 or D_2.

Now, suppose you have two domains D_1 and D_2, such that the intersection is nonempty and connected, and a function f_1 that is analytic over the domain D_1. If there exists a function f_2 that is analytic over the domain D_2 and such that $f_1 = f_2$ on the intersection $D_1 \cap D_2$, then we say f_2 is an *analytic continuation* of f_1 into domain D_2.

Now, whenever an analytic continuation exists, it is unique. The reason for this is a basic mathematical result from the theory of complex variables:

A function that is analytic in a domain D is uniquely determined over D by its values over a domain, or along an arc, interior to D.

Define the function $F(z)$, analytic over the union $D_1 \cup D_2$, as

$$F(z) \; = \; \left\{ \begin{array}{ll} f_1(z) & \text{when } z \text{ is in } D_1 \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{array} \right.$$

In other words, F is given by f_1 over D_1 and by f_2 over D_2, and since $f_1 = f_2$ over the intersection of D_1 and D_2, this is a well-defined, holomorphic function. By the mathematical result quoted above, since F is analytic in $D_1 \cup D_2$, it is uniquely determined by f_1 on D_1. (For that matter, it is also uniquely determined by f_2 on D_2.) In other words, there is only one possible holomorphic function on $D_1 \cup D_2$ that matches f_1 on D_1.

In this case, the function $F(z)$ is said to be the analytic continuation over $D_1 \cup D_2$ of either f_1 or f_2.

Example: Consider first the function

$$f_1(z) \; = \; \sum_{n=0}^{\infty} z^n$$

This power series converges when $|z| < 1$ to $1/(1-z)$, and is not defined when $|z| \geq 1$. (In particular, this is just a geometric series, so we can sum it as a geometric series, so long as we're in the region of convergence.)

On the other hand, the function

$$f_2(z) \;=\; \frac{1}{1-z}$$

is defined and analytic everywhere except $z = 1$.

Since $f_1 = f_2$ on the disk $|z| < 1$, we can view f_2 as the analytic continuation of f_1 to the rest of the complex plane (minus the point $z = 1$).

Example: Consider the function

$$f_1(z) \;=\; \int_0^\infty \exp(-zt)dt$$

This integral exists only when Re $z > 0$, and for such z, this integral has value $1/z$.

Since the function $1/z$ matches f_1 on the domain Re $z > 0$, the function $1/z$ is the analytic continuation of f_1 to nonzero complex numbers.

While we're at it, define

$$f_2(z) \;=\; i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n$$

This series converges for $|z+i| < 1$, and so f_2 is defined only within that disk centered on $-i$. Within that unit disk, one can show that $f_2(z) = 1/z$, using the fact that the series is a geometric series.

Since f_2 matches $1/z$ on a disk, we can view $1/z$ as the analytic continuation of f_2 to nonzero complex numbers.

Also, we can view f_2 as the analytic continuation of f_1 to the disk $|z+i| < 1$.

Example: The Gamma function.

Recall the second definition of the Gamma function,

$$\Gamma(z) \;=\; \int_0^\infty \exp(-t)t^{z-1}dt$$

is valid for Re $z > 0$. Other definitions, such as the Weierstrass form

$$\frac{1}{\Gamma(z)} \;=\; z\exp(\gamma z)\prod_{n=1}^{\infty}\left(1 \;+\; \frac{z}{n}\right)\exp(-z/n)$$

are valid more generally. Thus, we can view the Weierstrass form as an analytic continuation of the Euler integral form.

2 Exercises (taken from [1])

1. Show that the holomorphic function

$$f_2(z) \;=\; \frac{1}{z^2 \;+\; 1}$$

($z \neq \pm i$) is the analytic continuation of the function

$$f_1(z) \;=\; \sum_{n=0}^{\infty} (-)^n z^{2n}$$

($|z| < 1$) into the domain consisting of all points in the z plane except $z = \pm i$.

2. Show that the function $f_2(z) = 1/z^2$ ($z \neq 0$) is the analytic continuation of the function

$$f_1(z) \;=\; \sum_{n=0}^{\infty} (n \;+\; 1)(z \;+\; 1)^n$$

($|z+1| < 1$) into the domain consisting of all points in the z plane except $z = 0$.

3. Find the analytic continuation of the function

$$f(z) \;=\; \int_0^{\infty} t \exp(-zt) dt$$

(Re $z > 0$) into the domain consisting of all points in the z plane except the origin.

4. Show that the function $1/(z^2+1)$ is the analytic continuation of the function

$$f(z) \;=\; \int_0^{\infty} \exp(-zt)\,(\sin t)\; dt$$

(Re $z > 0$) into the domain consisting of all points in the z plane except $z = \pm i$.

References

[1] R. Churchill, J. Brown, *Complex Variables and Applications*, 5th edition, McGraw-Hill, 1990, section 102.