A proposal for $(0,2)$ *mirror symmetry of toric varieties*

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Based on Wei Gu, Eric Sharpe arXiv : 1707.05274

Contents

- Review of (2,2) mirror symmetry
- Proposal for (0,2) mirror symmetry
- Future directions

Mirror symmetry

- Worldsheet SUSY algebra is invariant under the outer automorphism given by the exchange of the generators

$$Q_- \leftrightarrow \overline{Q}_-, \qquad F_V \leftrightarrow F_A \tag{1}$$

- $Q_\pm$ and $\overline{Q}_\pm$ are SUSY generators, while F_V and F_A are the generators of vector R-symmetry and axial R-symmetry respectively.
- For CYs, mirror symmetry is a duality between interpretations of an SCFT. One implication: rotating the Hodge diamond, which for 3-folds acts as $H_1^{1,1} \Leftrightarrow H_2^{2,1}$.
- In this talk, we're going to mainly focus on mirror symmetry for Fano spaces, in which sigma model on Fano space mirror to a LG model, and in particular after reviewing (2,2), we'll discuss a proposal for (0,2).

A-twist and B-twist

A-twist

$$Q_A = \overline{Q}_+ + Q_-, \qquad M' = M + F_V.$$

B-twist

$$Q_B = \overline{Q}_+ + \overline{Q}_-, \qquad M' = M + F_A.$$

M is the lorentz generator. One can easily find that A-twist and B-twist are exchanged under mirror symmetry. In this talk, we mainly focus on twisted theory.

(2,2) Lagrangian for general toric variety

We consider a GLSM with gauge group $U(1)^k$, and N chiral superfields with gauge charge Q_i^a. We require $N > k$. The lagrangian can be written as

$$
\begin{aligned}
L &= \int d^4\theta \left(\sum_i \overline{\mathbf{\Phi}}_i e^{-Q_i^a V_a} \mathbf{\Phi}_i - \sum_a \frac{1}{e_a^2} \overline{\Sigma}_a \Sigma_a \right) + \int d^2\theta W(\mathbf{\Phi}) + c.c \\
&+ \int d^2\widetilde{\theta} \left(-\sum_a t_a \Sigma_a \right) + c.c.
\end{aligned}
$$

This is the classical action, the first two terms stand for the kinetic terms of the system, while W is the superpotential.

In the A-twist, the σ's are topological observables. (Morrison, Plesser '94).

Benini, Zaffaroni '15, Cyril, Cremonesi, Park '15 compute correlation functions for A-twisted GLSM in general by localization. The effective twisted superpotential

$$\widetilde{W}_{eff} = \sum_a \Sigma_a \left[-t_a + \sum_i Q_i^a \left(\log \left(\sum^a Q_i^a \Sigma_a \right) - 1 \right) \right].$$

$t_a(\mu) = t_a^{classical}(\Lambda_0) + \sum_i Q_i^a \log \frac{\mu}{\Lambda_0}$. μ is the physical scale which we set 1 in the previous slides while Λ_0 is the cutoff scale which for defining the theory at UV. The chiral ring relations are

$$\prod_i \left(\sum_a Q_i^a \sigma_a \right)^{Q_i^a} = q^a.$$

(2,2) Mirror

The dual theory is (Hori, Vafa '00)

$$
\begin{aligned}
L &= \int d^4\theta \left(-\sum_i (Y_i + \overline{Y}_i) \log \left(Y_i + \overline{Y}_i \right) - \sum_a \frac{1}{e_a^2} \overline{\Sigma}_a \Sigma_a \right) \\
&+ \int d^2\theta \sum_a \left(\sum_i Q_i^a Y_i - t^a \right) \Sigma_a + \sum_i e^{-Y_i} + c.c.
\end{aligned}
$$

The matter maps between two theories are

$$
Y_i + \overline{Y}_i = \overline{\mathbf{\Phi}}_i e^{-Q_i^a V_a} \mathbf{\Phi}_i
$$

In the B-twisted mirror, the TFT observable is y, the lowest component of Y. One can easily see that $x = e^{-y}$ is also a topological observable and used in B-model.

In B-model, all of the terms except superpotential are Q-exact, but in B-model superpotential will not receive quantum correction.

Mirror symmetry

For GLSM, now, we want to build more detailed maps between two sides.

$$
\begin{aligned}
Q_A, \quad F_V \quad &\Leftrightarrow \quad Q_B, \quad F_A \\
f(\sigma) \quad &\Leftrightarrow \quad g(e^{-y}) \\
\widetilde{W}_{eff} \quad &\Leftrightarrow \quad W \\
Correlation \quad functions \quad &= \quad Correlation \quad functions.
\end{aligned}
$$

The last three maps are only well-defined at the vacuum of the theories, while the first map can even define without at vacuum. The maps imply that if we know the detailed map between the σ and e^{-y}, we can obtain the mirror side's superpotential and compare the correlation functions between two sides.

Ansatz for the map between observable of two sides

The D-term equations are the following constraints

$$\sum_i Q_i^a y_i - t^a = 0,$$

then

$$\prod_i \left(e^{-y_i}\right)^{Q_i^a} = q^a.$$

We assume the above are the chiral ring relations for B-model, because the chiral ring of A-model is $\prod_i \left(\sum_a Q_i^a \sigma_a\right)^{Q_i^a} = q^a$, then the ansatz of operator mirror map is

$$\sum_a Q_i^a \sigma_a \Leftrightarrow e^{-y_i}$$

Sometimes people write the lowest y as superfield Y without causing any confusion.

One comment: one can use above to re-derive the terms $\sum_i e^{-Y_i}$ should appear in the B-model LG superpotential.

A-model correlation function

A-model correlation function. Melnikov, Plesser '05, Cyril, Cremonesi, Park '15, Nekrasov, Shatashvili '14.

$$\langle \mathcal{O}(\sigma) \rangle \cong \sum_{\widehat{\sigma}_v \in V} \frac{\mathcal{O}(\widehat{\sigma}_v) \mathcal{Z}_0^{1-loop}(\widehat{\sigma}_v)}{H(\widehat{\sigma}_v)}.$$

$H(\widehat{\sigma}) = \det_{ab}\left(\partial_{\sigma_a}\partial_{\sigma_b}\widetilde{W}_{eff}\right)$. The V denotes the solutions of

$$\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a} = 0, \qquad for \qquad a = 1, \cdots, k,$$

and

$$\mathcal{Z}_0^{1-loop}(\widehat{\sigma}) = \prod_i \left(\sum^a Q_i^a \sigma_a \right)^{r_i - 1}.$$

r_i is the R-charge for the chiral superfield $\mathbf{\Phi}_i$.

B twisted LG correlation function

For B-model, the LG correlation functions were obtained much earlier by Vafa , which are

$$\langle \mathcal{O}(X) \rangle = \sum_{V} \frac{\mathcal{O}(X_v)}{H(X_v)},$$

where $X = e^{-Y}$, and $H(X_v) = \det(\partial_{\Theta_A} \partial_{\Theta_B} W)$. The Θ_A are the fundamental variables in B-model and their detailed meaning are following

$$Y_i = \sum_{A} V_i^A \Theta_A + t_i.$$

Solving D-term constraints $\sum_i Q_i^a Y_i - t^a = 0$, we have

$$\sum_{i} Q_i^a V_i^A = 0, \qquad \sum_{i} Q_i^a t_i = t^a.$$

Example: $\mathbb{CP}^4$

The charge matrix is

$$Q_i^a = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

then V_i^A is

$$V_i^A = \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \end{pmatrix}.$$

$\mathbb{CP}^4$

The A-model twisted superpotential and B-model superpotential are

$$\widetilde{W}_{eff} = -t\sigma + 5\sigma\left(\log\sigma - 1\right), \qquad W = \sum_{i=1}^{5} e^{-Y_i}$$

Chiral ring relations are

$$\sigma^5 = q, \qquad X^5 = q$$

There are five vacua. One can also compute that

$$H(\sigma)\prod_{i=1}^{5} Q_i^a \sigma_a = 5\sigma^4, \qquad H(X) = 5X^4$$

The correlation functions are

$$\langle\sigma^{5k+4}\rangle = q^k, \qquad \langle X^{5k+4}\rangle = q^k, \qquad \textit{for} \quad k \geq 0.$$

The correlation functions match

We can prove the correlation functions match in general under the observable map

$$\mathcal{O}(\sigma) \Leftrightarrow \mathcal{O}(X)$$

$$H(\sigma)\prod_i \left(\sum_a Q_i^a \sigma_a\right) \Leftrightarrow H(X)$$

The first map is about the match of chiral ring relations, and written the second one in details:

$$\det_{ab}\left(\sum_i \frac{Q_i^a Q_i^b}{\left(\sum_a Q_i^a \sigma_a\right)}\right) \prod_i \left(\sum_a Q_i^a \sigma_a\right) = \det_{AB}\left(\sum_{i,a} V_i^A V_i^B \left(Q_i^a \sigma_a\right)\right)$$

We have used the operator mirror map $\sum_a Q_i^a \sigma_a \Leftrightarrow e^{-y_i}$. The above has been proved in Gu, Sharpe '17.

Basics of (0,2) superfields

The bose $(0,2)$ chiral superfield Φ, in some representation of the gauge group, satisfying

$$\overline{\mathcal{D}}_+\Phi = 0.$$

Its θ expansion is

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\phi.$$

Here $\mathcal{D}_\alpha$ is now the gauge-covariant derivatives at $\theta^+ = \overline{\theta}^+ = 0$.

Basics of (0,2) superfields

Fermi multiplets: anticommuting, negative chirality spinor super field Λ_-, in some representation of the gauge group, obeying

$$\overline{\mathcal{D}}_+ \Lambda_- = \sqrt{2} E,$$

where E is some superfield satisfying

$$\overline{\mathcal{D}}_+ E = 0.$$

The θ expansion of the fermi multiplet is

$$\Lambda_- = \lambda_- - \sqrt{2}\theta^+ G - i\theta^+ \overline{\theta}^+ (\mathcal{D}_0 + \mathcal{D}_1)\lambda_- - \sqrt{2}\overline{\theta}^+ E.$$

E is a holomorphic function of chiral superfields Φ_i, it has Θ expansion

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+ \frac{\partial E}{\partial \phi_i} \psi_{+,i} - i\theta^+ \overline{\theta}^+ \left(\mathcal{D}_0 + \mathcal{D}_1\right) E(\phi_i).$$

(0,2) mirror symmetry

(0,2) GLSM

We consider $A/2$ twisted abelian GLSM which is a deformation of a (2,2) theory.

$$
\begin{aligned}
L &= L_{gauge} + L_{ch} + L_F + L_{D,\theta} + L_J \\
&= \frac{1}{2e^2}\sum_a \int d\theta^+ d\overline{\theta}^+ \overline{\Upsilon}_a \Upsilon_a \\
&- \sum_i \frac{i}{2}\int d^2\theta \overline{\Phi}_i \Phi_i - \sum_j \frac{1}{2}\int d^2\theta \overline{\Lambda}_{-,j}\Lambda_{-,j} \\
&\quad + \left(\int d\theta^+ \sum_a \frac{t_a}{2}\Upsilon_a \mid_{\overline{\theta}^+=0} + c.c\right) - \left(\frac{1}{\sqrt{2}}\int d\theta^+ \left(\sum_j \Lambda_{-,j} J^j\right)\right)
\end{aligned}
$$

The appearance of function J^i is related to the construction of hypersurface in $(0,2)$ model, and it has the constraint that

$$\sum_i E_i J^i = 0.$$

On Coulomb branch, we have

$$E_I = \sum_{a=1}^{k} \sigma_a E_I^a(\Phi),$$

for some holomorphic functions $E_I^a(\Phi)$, and the matter multiplets Φ_I, $\Lambda_{-,I}$ acquire masses

$$M_{IJ} = \partial_J E_I \mid_{\phi=0} = \sum_{a=1}^{k} \sigma_a \partial_J E_I^a \mid_{\phi=0} .$$

Generic point of coulomb branch

$$t^a_{\mathit{eff}} = t^a - \sum_\alpha Q^a_\alpha \log \left(\det M_\alpha\right).$$

$\sum_\alpha k_\alpha = N$. The vacuum of $A/2$-model thus corresponds to

$$t^a_{\mathit{eff}} = \frac{\partial \widetilde{W}_{\mathit{eff}}}{\partial \Upsilon_a} = 0,$$

then we have

$$\prod_\alpha \left(\det M_\alpha\right)^{Q^a_\alpha} = q^a, \qquad \mathit{where} \quad q^a = e^{-t^a}.$$

McOrist, Melnikov '08.

$\mathbb{P}^1 \times \mathbb{P}^1$

The most general possible (0,2) deformation is

$$M = A\sigma + B\widetilde{\sigma}, \qquad \widetilde{M} = C\sigma + D\widetilde{\sigma},$$

where A, B, C, D are two by two matrices and for $\mathcal{N} = (2,2)$ locus $A = D = I$ and $B = C = 0$. The chiral ring relations are

$$\det(A\sigma + B\widetilde{\sigma}) = q_1, \qquad \det(C\sigma + D\widetilde{\sigma}) = q_2$$

We will return to this example later.

Any B/2 LG model has the form

$$
\begin{aligned}
L &= \int d^2\theta K\left(Y_i, \overline{Y}_i, F_i, \overline{F}_i\right) + K\left(\Sigma_a, \overline{\Sigma}_a, \Upsilon_a, \overline{\Upsilon}_a\right) \\
&+ \left(-\int d\theta^+ \sum_{a=1}^{k}\sum_{i=1}^{N} W + c.c\right).
\end{aligned}
$$

The Kahler potentials of B/2 LG model are Q-exact and do not contribute the correlation function, so we do not write them out explicitly.

A very general expression for mirror W was proposed by Adams et.al '03

$$
W = -\sum_{a=1}^{k}\sum_{i=1}^{N} \frac{i\Upsilon^a}{2}\left(Q_i^a Y_i - t^a\right) + \sum_a \Sigma_a F_a + \sum_{i,j} \beta_{ij} F^i e^{-Y_j},
$$

where β_{ij} are some parameters and $\beta_{ij} = -\delta_{ij}$ is (2,2) theory.

Toric Deformation and mirror map of the observable

We focus on the subclass of toric deformation in A/2 model, meaning:

$$E_i = \sum_{a,j} A_{ij} Q_j^a \sigma_a \phi_i.$$

$A_{ij} = \delta_{ij} + B_{ij}$ obeys the following constraint. Choose a $k \times k$ square submatrix $S \subseteq (Q_i^a)$ of rank=k and restrict to deformations B_{ij} such that $B_{ij} = 0, \quad \text{for} \quad i \in S$. Chiral ring relations: $\prod_i \left(\sum_{a,j} A_{ij} Q_j^a \sigma_a\right)^{Q_i^a} = q^a$.

For example for $\mathbb{P}^1 \times \mathbb{P}^1$, we choose the first column and the third column of charge matrix as the submatrix S which is identity.

$$M = \begin{pmatrix} \sigma & \\ & c\sigma + d\widetilde{\sigma} \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \widetilde{\sigma} & \\ & g\widetilde{\sigma} + h\sigma \end{pmatrix}.$$

Adams et.al '05, Closset et.al '15 indicate that σ and Y still can be treated as topological observable in (0,2) theory with (2,2) locus.

Solving the D-term for (0,2) B-model LG as in Adams et.al '03

$$\sum_i Q_i^a Y_i = t^a,$$

then

$$\prod_i \left(e^{-Y_i}\right)^{Q_i^a} = q^a,$$

Looks same as (2,2) case. Follow the (2,2) case, the ansatz of operator mirror map we suggest is

$$\sum_{a,j} A_{ij} Q_j^a \sigma_a \Leftrightarrow e^{-Y_i}.$$

(2,2) mirror in (0,2) language

The (2,2) B-model superpotential can be written into (0,2) form, which is

$$W = -\sum_{a=1}^{k}\sum_{i=1}^{N}\frac{i\Upsilon^a}{2}\left(Q_i^a Y_i - t^a\right) + \sum_a \Sigma_a F_a - \sum_i F_i e^{-Y_i}.$$

After integrated out the fields Υ and Σ, we will get the following equations.

$$\sum_i Q_i^a Y_i = t^a, \qquad \sum_i Q_i^a F_i = 0.$$

We can solve it as the following expression

$$Y_i = \sum_A V_i^A \Theta_A + t_i, \qquad F_i = \sum_A V_i^A G_A,$$

where $A = 1, \cdots, N-k$ and $\sum_i Q_i^a V_i^A = 0$ Then the final expression of the dual twisted superpotential is

$$W = -\sum_{A,i} G_A \left(V_i^A e^{-Y_i}\right).$$

Consider (0,2) deformation of the previous result

$$W' = -\sum_A G_A \left(\sum_i V_i^A e^{-Y_i} + D_{i_l}^A \left(e^{-Y_{i_l}} \right) \right)$$

where the index i_l corresponds to the index we choose for square submatrix $RankQ_{i_l}^a = k$. The vacuum equation $\partial W / \partial G_A = 0$ should reproduce the mirror map of the observable which restricts the expression of D_{i_l}. By plugging in the mirror map $\sum_{a,j} A_{ij} Q_j^a \sigma_a \Leftrightarrow e^{-Y_i}$, we get

$$\sum_{i,j} V_i^A B_{ij} Q_j^a + \sum_{i_l} D_{i_l}^A Q_{i_l}^a = 0, \tag{2}$$

where the index i_l corresponds to the index we choose for sub square-matrix $RankQ_{i_l}^a = k$, then

$$D_{i_l}^A = -\sum_{i,j} \sum_a V_i^A B_{ij} Q_j^a [Q^{-1}]_{ai_l}. \tag{3}$$

The number of constraint equations implies why we only consider the subclass of toric deformation.

Correlation functions match

- A/2 model. McOrist, Melnikov '08, Closset, Gu, Jia, Sharpe '15.

$$\langle \mathcal{O}(\sigma) \rangle = \frac{\mathcal{O}(\sigma) \mathcal{Z}^{1-loop}}{H(\sigma)} |_{vacuum}$$
$$= \frac{\mathcal{O}(\sigma)}{\det \left(\sum_i \frac{\sum_j Q_i^a A_{ij} Q_j^b}{\sum_n A_{in} Q_n^c \sigma_c} \right) \prod_i \left(\sum_k A_{ik} Q_k^d \sigma_d \right)^{1-r_i}} |_{vacuum}$$

- For B/2-model. Melnikov '09

$$\langle \mathcal{O}(X) \rangle = -\frac{\mathcal{O}(X)}{\det \partial_{G_A} \partial_{\Theta_B} W} |_{vacuum}$$

One can follow almost the same procedures in (2,2) case to prove that the correlation function of A/2 model is same as B/2.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

The charge matrix is

$$Q_i^a = \begin{pmatrix} 1 & 1 & & \\ & & 1 & 1 \end{pmatrix},$$

and the dual matrix can be solved as

$$V_i^A = \begin{pmatrix} 1 & -1 & & \\ & & -1 & 1 \end{pmatrix}.$$

the deformation we consider here is

$$E_1 = \sigma\phi_1, \qquad E_2 = (\sigma + \epsilon_2\widetilde{\sigma})\,\phi_2, \qquad E_3 = \widetilde{\sigma}\phi_3, \qquad E_4 = (\widetilde{\sigma} + \epsilon_4\sigma)\,\phi_4.$$

Thus the chiral ring relations of $A/2$-model are

$$\sigma\,(\sigma + \epsilon_2\widetilde{\sigma}) = q_1, \qquad \widetilde{\sigma}\,(\widetilde{\sigma} + \epsilon_4\sigma) = q_2.$$

From previous slide, we see that the submatrix S we pick is the first column and the third

$$S = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon_4 & 0 & 0 & 0 \end{pmatrix}$$

and from the mirror asnatz, we obtain the following Toda dual of superpotential,

$$W = F_1\left(e^{-Y_1} - q^1 e^{Y_1}\right) + F_3\left(e^{-Y_3} - q^2 e^{Y_3}\right) + \epsilon_2 F_1 e^{-Y_3} + \epsilon_4 F_3 e^{-Y_1}.$$

We define Θ, $\widetilde{\Theta}$ as

$$Y_1 = \Theta, \qquad Y_2 = t_1 - \Theta, \qquad G^1 = F_1 = -F_2,$$

and

$$Y_3 = \widetilde{\Theta}, \qquad Y_4 = t_2 - \widetilde{\Theta}, \qquad G^2 = -F_3 = F_4.$$

$\mathbb{P}^1 \times \mathbb{P}^1$

Let us define the low energy theory in terms of single-valued degrees of freedom $X = e^{-\Theta}$ and $\widetilde{X} = e^{-\widetilde{\Theta}}$. Then the chiral ring relations are

$$X(X + \epsilon_2 \widetilde{X}) = q_1, \qquad \widetilde{X}(\widetilde{X} + \epsilon_4 X) = q_2$$

which agrees with the A/2-model chiral ring relations.
The classical correlation functions of A/2-model are

$$\langle \sigma\sigma \rangle = -\frac{\epsilon_2}{1 - \epsilon_2\epsilon_4}, \qquad \langle \sigma\widetilde{\sigma} \rangle = 1, \qquad \langle \widetilde{\sigma}\widetilde{\sigma} \rangle = -\frac{\epsilon_4}{1 - \epsilon_2\epsilon_4}.$$

The Hessian factor of B/2-model is

$$\begin{aligned} \det \frac{\partial^2 W'}{\partial G_A \partial \Theta_B} &= \begin{pmatrix} e^{-Y_1} + e^{-Y_2} & \epsilon_2 e^{-Y_3} \\ \epsilon_4 e^{-Y_1} & e^{-Y_3} + e^{-Y_4} \end{pmatrix} \\ &= \left(4X\widetilde{X} + 2\epsilon_2 \widetilde{X}^2 + 2\epsilon_4 X^2\right), \end{aligned}$$

where we have plugged in $X = e^{-Y_1}$, $\widetilde{X} = e^{-Y_3}$ as well as $X + \epsilon_2 \widetilde{X} = e^{-Y_2}$, $\widetilde{X} + \epsilon_4 X = e^{-Y_4}$.

$\mathbb{P}^1 \times \mathbb{P}^1$

The classical correlation functions for B/2-model are

$$\langle X^2 \rangle = -\frac{\epsilon_2}{1-\epsilon_2\epsilon_4}, \qquad \langle X\widetilde{X} \rangle = 1, \qquad \langle \widetilde{X}^2 \rangle = -\frac{\epsilon_4}{1-\epsilon_2\epsilon_4}.$$

By combining the chiral ring relations, we see that the B/2-model is a mirror map of A/2-model.

Zhuo et.al '16 guessed the mirror ansatz of $\mathbb{P}^1 \times \mathbb{P}^1$ with the general deformation, and our calculation here agrees with theirs when restricted to the deformation considered here.

Hirzebruch surface

The $A/2$-model theory has charge matrix for Hirzebruch surface is

$$Q_i^a = \begin{pmatrix} 1 & & 1 & n \\ & 1 & & 1 \end{pmatrix},$$

and (0,2) theory with (2,2) locus we consider has the following deformation (we have picked the S, which will show later)

$$\begin{aligned} E_1 &= \sigma\phi_1, \qquad E_2 = \widetilde{\sigma}\phi_2, \\ E_3 &= (\sigma + \epsilon_3\widetilde{\sigma})\,\phi_3, \qquad E_4 = (n\sigma + \widetilde{\sigma} + \epsilon_4\sigma)\,\phi_4. \end{aligned}$$

Its chiral ring relations are

$$\sigma\,(\sigma + \epsilon_3\widetilde{\sigma})\,((n+\epsilon_4)\sigma + \widetilde{\sigma})^n = q_1, \qquad \widetilde{\sigma}\,((n+\epsilon_4)\sigma + \widetilde{\sigma}) = q_2.$$

The dual matrix is chosen to be

$$V_i^A = \begin{pmatrix} -1 & & 1 & \\ -n & -1 & & 1 \end{pmatrix}.$$

The submatrix S we pick is the first and the second column of the charge matrix.

$$S = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

and the deformation is

$$B_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon_3 & 0 & 0 \\ \epsilon_4 & 0 & 0 & 0 \end{pmatrix}.$$

Follow the mirror ansatz , we can derive the $B/2$-model twisted superpotential

$$W = -\left[F_1 e^{-Y_1} + F_2 e^{-Y_2} + F_3 e^{-Y_3} + F_4 e^{-Y_4} - \epsilon_3 F_3 e^{-Y_2} - \epsilon_4 F_4 e^{-Y_1}\right].$$

The chiral ring relations for $B/2$-model is

$$X\left(X + \epsilon_3 \widetilde{X}\right)\left((n+\epsilon_4)X + \widetilde{X}\right)^n = q_1, \qquad \widetilde{X}\left((n+\epsilon_4)X + \widetilde{X}\right) = q_2.$$

With variable map as

$$\begin{aligned} \sigma \Leftrightarrow X &= e^{-Y_1}, \quad \sigma + \epsilon_3 \widetilde{\sigma} \Leftrightarrow \left(X + \epsilon_3 \widetilde{X}\right) = e^{-Y_3}, \quad \widetilde{\sigma} \Leftrightarrow \widetilde{X} \quad (4) \\ &= e^{-Y_2}, \quad (n+\epsilon_4)\,\sigma + \widetilde{\sigma} \Leftrightarrow \left((n+\epsilon_4)\,X + \widetilde{X}\right) = e^{-Y_4}. \end{aligned}$$

The determinant Hessian with one-loop factor in $A/2$-model

$$\frac{H}{\mathcal{Z}^{1-loop}} = \det\begin{pmatrix} \frac{2\sigma+\epsilon_3\widetilde{\sigma}}{\sigma(\sigma+\epsilon_3\widetilde{\sigma})} + \frac{n(n+\epsilon_4)}{(n+\epsilon_4)\sigma+\widetilde{\sigma}} & \frac{n+\epsilon_4}{(n+\epsilon_4)\sigma+\widetilde{\sigma}} \\ \frac{n}{\sigma+\epsilon_3\widetilde{\sigma}} + \frac{n}{(n+\epsilon_4)\sigma+\widetilde{\sigma}} & \frac{2\widetilde{\sigma}+(n+\epsilon_4)\sigma}{\widetilde{\sigma}((n+\epsilon_4)\sigma+\widetilde{\sigma})} \end{pmatrix}^T \tag{5}$$
$$\times\sigma\left(\sigma+\epsilon_3\widetilde{\sigma}\right)\widetilde{\sigma}\left((n+\epsilon_4)\sigma+\widetilde{\sigma}\right)$$
$$= \left(4+n(n+\epsilon_4)\epsilon_3\right)\sigma\widetilde{\sigma} + (n+\epsilon_4)(n+2)\sigma^2 + 2\epsilon_3\widetilde{\sigma}^2$$

Similarly the determinant of $B/2$-model Hessian expression is following

$$\det\begin{pmatrix} e^{-Y_1}+e^{-Y_3} & ne^{-Y_1}+\epsilon_3 e^{-Y_2} \\ ne^{-Y_1}+\epsilon_4 e^{-Y_1} & n^2 e^{-Y_1}+e^{-Y_2}+e^{-Y_4}+n\epsilon_4 e^{-Y_1} \end{pmatrix} \tag{6}$$
$$= \left(4+n(n+\epsilon_4)\epsilon_3\right)X\widetilde{X} + (n+\epsilon_4)(n+2)X^2 + 2\epsilon_3\widetilde{X}^2,$$

We can easily find that (5) and (6) are same under the map $\sigma \leftrightarrow X$, $\widetilde{\sigma} \leftrightarrow \widetilde{X}$, thus correlation functions are same.

dP_2

Zhuo et.al '16 '17 guessed what the mirror ansatz of some examples look like. Our calculations agree with theirs, and furthermore our ansatz can be used to other toric varieties' study as well. We are going to use dP_2 to show how our predictions match with Zhuo et.al '17

dP_2

The charge matrix of the GLSM for the chiral fields of dP_2 is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

We will take

$$(V_i^A) = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}.$$

For (2,2), we have $E_i = \sum_{a=1}^{3} Q_{ia}\sigma^a\phi_i$. Follow the reference, we use the following notations: $Q_{1a} = \alpha_a$, $Q_{2a} = \beta_a$, $Q_{3a} = \gamma_a$, $Q_{4a} = \delta_a$, and $Q_{5a} = \epsilon_a$. For (0,2) deformation, the α.. can have some more general values and we will see this later.

First choice of S

Pick the first, third, and fifth columns of the charge matrix be the S

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

Thus, the deformation we are considering is

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & 1 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The (0,2) deformation we consider is

$$E_i = \sum_{a,j} A_{ij} Q_j^a \sigma_a \phi_i,$$

which can be written in detail as

$$\begin{aligned}
E_1 &= (\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\phi_1 \;=\; (\sigma_1 + \sigma_3)\phi_1, \\
E_2 &= (\beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3)\phi_2, \\
&= (A_{21}(\sigma_1 + \sigma_3) + A_{22}\sigma_1 + A_{23}(\sigma_1 + \sigma_2) + A_{24}\sigma_2 + A_{25}\sigma_3)\,\phi_2, \\
E_3 &= (\gamma_1\sigma_1 + \gamma_2\sigma_2 + \gamma_3\sigma_3)\phi_3 \;=\; (\sigma_1 + \sigma_2)\phi_3, \\
E_4 &= (\delta_1\sigma_1 + \delta_2\sigma_2 + \delta_3\sigma_3)\phi_4, \\
&= (A_{41}(\sigma_1 + \sigma_3) + A_{42}\sigma_1 + A_{43}(\sigma_1 + \sigma_2) + A_{44}\sigma_2 + A_{45}\sigma_3)\,\phi_4, \\
E_5 &= (\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3)\phi_5 \;=\; \sigma_3\phi_5,
\end{aligned}$$

from which we find

$$\begin{aligned}
\vec{\alpha} &= (1,0,1), \quad \vec{\gamma} = (1,1,0), \quad \vec{\epsilon} = (0,0,1), \\
\vec{\beta} &= (A_{21} + A_{22} + A_{23}, A_{23} + A_{24}, A_{21} + A_{25}) \\
\vec{\delta} &= (A_{41} + A_{42} + A_{43}, A_{43} + A_{44}, A_{41} + A_{45}).
\end{aligned}$$

Mirror

Follow the mirror ansatz, we have

$$(D^A_{i_S}) = \begin{bmatrix} \begin{matrix} A_{21}+A_{22} \\ -1-A_{24} \end{matrix} & A_{23}+A_{24} & \begin{matrix} -A_{22}+A_{24} \\ +A_{25}+1 \end{matrix} \\ \begin{matrix} A_{21}+A_{22}-A_{24}+ \\ A_{41}+A_{42}-A_{44} \end{matrix} & \begin{matrix} A_{23}+A_{24}+ \\ A_{43}+A_{44}-1 \end{matrix} & \begin{matrix} -A_{22}+A_{24}+A_{25} \\ -A_{42}+A_{44}+A_{45} \end{matrix} \end{bmatrix}$$

The proposed mirror superpotential $-W$ is

$$G_1 \Big[(A_{21}+A_{22}-A_{24})X_1 - \frac{q_1}{X_1X_3} + (A_{23}+A_{24})X_3 \\ +(-A_{22}+A_{24}+A_{25})X_5] + \\ G_2 \Big[(A_{21}+A_{22}-A_{24}+A_{41}+A_{42}-A_{44})X_1 - \frac{q_1}{X_1X_3} - \frac{q_2}{X_3} + A_{23}X_3 \\ (A_{24}+A_{43}+A_{44})X_3 + X_5(A_{24}-A_{22}+A_{25}-A_{42}+A_{44}+A_{45})] \,,$$

Now, let us compare to the first (0,2) mirror proposal for dP_2 in Zhuo et.al '17. In their notation

$$\alpha \cdot X = X_1, \quad \gamma \cdot X = X_3, \quad \epsilon \cdot X = X_5,$$

$$\begin{aligned}
\beta \cdot X &= (A_{21} + A_{22} - A_{24})X_1 + (A_{23} + A_{24})X_3 + (A_{24} + A_{25} - A_{22})X_5 \\
\delta \cdot X &= (A_{41} + A_{42} - A_{44})X_1 + (A_{43} + A_{44})X_3 + (A_{44} + A_{45} - A_{22})X_5
\end{aligned}$$

and

$$\begin{aligned}
J_1 &= -\frac{q_1}{X_1 X_3} + Z\frac{q_3}{X_1 X_5} + X_5 + \beta \cdot X, \quad J_Z = 1 - \frac{q_3}{X_1 X_5}, \\
J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1 X_3} + \beta \cdot X + \delta \cdot X \quad J_5 = X_5 + Z\frac{q_3}{X_1 X_5},.
\end{aligned}$$

Solving $J_Z = J_5 = 0$, we get $-Z = X_5 = q_3/X_1$. Left two J's:

$$\begin{aligned}
J_1 &= -\frac{q_1}{X_1 X_3} + (A_{21} + A_{22} - A_{24})X_1 + (A_{23} + A_{24})X_3 \\
&\quad +(A_{24} + A_{25} - A_{22})\frac{q_3}{X_1}, \\
J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1 X_3} + (A_{21} + A_{22} - A_{24} + A_{41} + A_{42} - A_{44})X_1 \\
&\quad +(A_{23} + A_{24} + A_{43} + A_{44})X_3
\end{aligned}$$

Second choice of S

Take the second, fourth, and fifth columns of the charge matrix, so that S is the identity. The allowed deformations are

$$(A_{ij}) = \left[\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & 1 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

To find the corresponding bundle deformation parameters, we compare the E's:

$$\begin{array}{rcl} E_1 & = & (\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\phi_1, \\ & = & (A_{11}(\sigma_1+\sigma_3) + A_{12}\sigma_1 + A_{13}(\sigma_1+\sigma_2) + A_{14}\sigma_2 + A_{15}\sigma_3)\,\phi_1, \\ E_2 & = & (\beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3)\phi_2 \;=\; \sigma_1\phi_2, \\ E_3 & = & (\gamma_1\sigma_1 + \gamma_2\sigma_2 + \gamma_3\sigma_3)\phi_3, \\ & = & (A_{31}(\sigma_1+\sigma_3) + A_{32}\sigma_1 + A_{33}(\sigma_1+\sigma_2) + A_{34}\sigma_2 + A_{35}\sigma_3)\,\phi_3, \\ E_4 & = & (\delta_1\sigma_1 + \delta_2\sigma_2 + \delta_3\sigma_3)\phi_4 \;=\; \sigma_2\phi_4, \\ E_5 & = & (\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3)\phi_5 \;=\; \sigma_3\phi_5, \end{array}$$

We conclude

$$\vec{\alpha} = (A_{11}+A_{12}+A_{13}, A_{13}+A_{14}, A_{11}+A_{15}),$$
$$\vec{\beta} = (1,0,0), \quad \vec{\delta} = (0,1,0), \quad \vec{\epsilon} = (0,0,1),$$
$$\vec{\gamma} = (A_{31}+A_{32}+A_{33}, A_{33}+A_{34}, A_{31}+A_{35}).$$

From our mirror ansatz, we have

$$(D^A_{i_S}) = -\left[\begin{array}{ccc} A_{11}+A_{12}+A_{13}-1 & A_{13}+A_{14} & A_{11}+A_{15}-1 \\ A_{31}+A_{32}+A_{33}-1 & A_{33}+A_{34}-1 & A_{31}+A_{35} \end{array}\right],$$

then the proposed mirror superpotential $-W$ is

$$G_1\left(\frac{q_1}{q_2}\frac{X_4}{X_2} - (A_{11}+A_{12}+A_{13})X_2 - (A_{13}+A_{14})X_4 - (A_{11}+A_{15})X_5\right)$$
$$+G_2\left(\frac{q_2}{X_4} - (A_{31}+A_{32}+A_{33})X_2 - (A_{33}+A_{34})X_4 - (A_{31}+A_{35})X_5\right)$$

where $X_i = \exp(-Y_i)$. It agrees with the previous study.

Conclusions

- We briefly reviewed the (2,2) mirror symmetry with some new aspects.
- We presented a mirror proposal for (0,2).

Future works

- Can we express the (0,2) twisted correlation function for higher genus?
- Can we prove that the toric deformation is the whole story?
- Can we extend our works to non-abelian cases?
- Can we find the general (0,2) theories' mirror ansatz?