Topological defects and decomposition

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Based on several papers with L. Ling, T. Pantev, E. Sharpe, and T. Vandermeulen.

A modern perspective on symmetry

Ordinary symmetries in QFT $\longleftrightarrow$ Topological domain walls

As an example, if we have a one-parameter continuous symmetry, then Noether gives us a closed $(d-1)$-form j. Then to any codimension one submanifold M of spacetime, we can construct a one-parameter family of operators

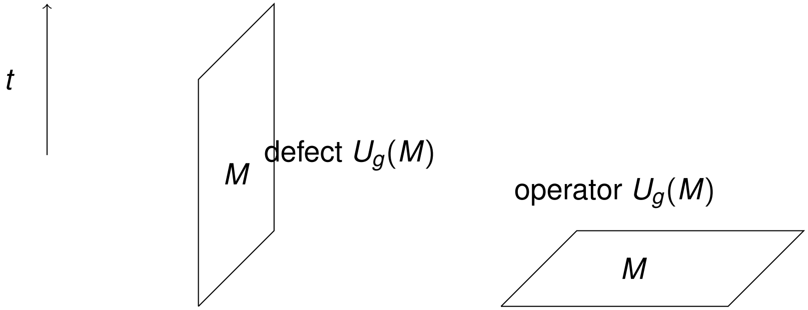
$$U_\alpha(M) = \exp\left[i\alpha \int_M j\right].$$

These can be inserted in correlation functions, and $dj = 0$ implies that we can deform M without changing the value of the correlation function - the operators are **topological**.

It works with discrete symmetries too - define the operator by its implementation of the symmetry.

Operators and defects

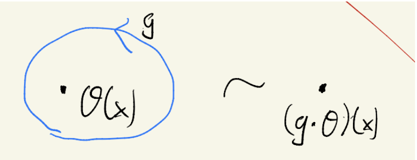
When the submanifold M wraps a spatial slice, we should think of $U_g(M)$ as an operator; when M is extended in the time direction it would be more natural to call it a defect.



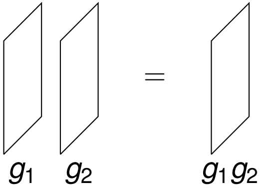
In this talk we will be sticking to Euclidean signature, in which case the distinction is not meaningful.

Realizing the symmetry group

In a correlation function, if we have $U_g(S^{D-1})$ surrounding a local operator $\mathcal{O}(x)$, this is equivalent to $(g \cdot \mathcal{O})(x)$.



If we bring two defects together (e.g. parallel Σ_{D-1} slices in $\Sigma_{D-1} \times \mathbb{R}$), they can be fused into a single defect according to the group law.



Generalizations

Any familiar aspect of symmetry in QFT can be translated into this language.

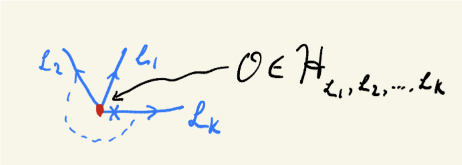
But there is a bonus! Now we have two natural ways to generalize the story[1].

- We can look at higher codimension topological defects. A co-dimension $p+1$ topological defect is called a "*p*-form" symmetry. It acts in a natural way on *p*-dimensional extended objects in the theory.
- We can look at the full collection of topological operators in the theory. These will have some fusion algebra, but it may not be group-like - there can be "non-invertible symmetries".

[1] Not new - just... very natural, and in a unified framework.

2D CFT with ordinary symmetry

Our main example will be 2D CFT with a finite global symmetry group, say G. The topological lines are labeled by elements $g \in G$, and they fuse according to the group multiplication. These lines can meet at junctions where we insert some junction operator from a Hilbert space $\mathcal{H}_{g_1,\cdots,g_k}$.



Different cyclic orderings of the lines correspond to non-canonically isomorphic Hilbert spaces.

If the operator at the junction has weight $h = \bar{h} = 0$, then the junction can also be moved around inside correlation functions without changing the result; it is a topological junction.

Topological junctions and OPEs

We use $V_{g_1,\cdots,g_n}$ to denote the weight zero subspace of $\mathcal{H}_{g_1,\cdots,g_n}$.

For the case of effective group-like symmetries, these topological junctions are characterized by

$$\dim_{\mathbb{C}}(V_{g_1,\cdots,g_n}) = \begin{cases} 1, & \text{if } g_1 \cdots g_n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

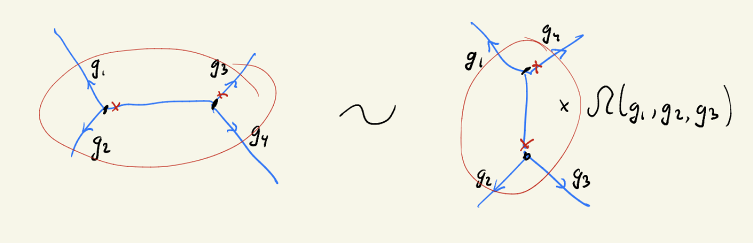
Just as with bulk operators, we can take OPEs of junction operators, which can be interpreted as a unitary map

$$\mathcal{H}_{g_1,\cdots,g_{n-1},g} \otimes \mathcal{H}_{g^{-1},h_1,\cdots,h_{m-1}} \longrightarrow \mathcal{H}_{g_1,\cdots,g_{n-1},h_1,\cdots,h_{m-1}}.$$

This OPE also restricts naturally to the topological spaces $V_{g_1,\cdots,g_n}$.

Anomaly

The OPE means that each way of resolving a topological 4-junction into a pair of topological 3-junctions can be represent by a net phase. Since $g_1 g_2 g_3 g_4 = 1$, the phase depends on three independent group elements, $\omega(g_1, g_2, g_3)$. It is a 3-chain for the group G called the **anomaly**.



Anomaly continued

Shifting the choice of basis vector in each $V_{g_1,g_2,(g_1g_2)^{-1}}$ multiplies $\omega(g_1, g_2, g_3)$ by a phase $\lambda(g_2, g_3)\lambda(g_1, g_2g_3)/\lambda(g_1, g_2)\lambda(g_1g_2, g_3)$. As well, there's a pentagon identity

$$\omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) = \omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4).$$

This says that ω is co-closed and can be shifted by co-exact things; it is classified by the group cohomology $H^3(G, U(1))$.

Networks of lines and gauging

A network of lines $\longleftrightarrow$ A background gauge configuration

(E.g. pick a triangulation of your worldsheet and put lines on each of your edges. The lines tell you what symmetry transformation to apply as you go between faces.)

Gauging $\longleftrightarrow$ Summing over background configurations

Equivalently, we can formally insert the line $\bigoplus_{g \in G} g$ on all edges of some sufficiently fine triangulation.

Discrete torsion, effect of anomaly

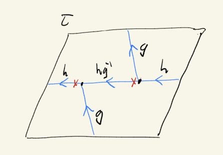
Of course when summing over networks, we need to pick junction vectors to put at the vertices. For a “local” action for the gauge field, this choice should only depend on the lines g_1, g_2, and $(g_1 g_2)^{-1}$ at the vertex, which can be represent by a 2-cochain $\alpha(g_1, g_2)$. Compatibility with a given choice of ω requires that $d\alpha = 1$, while a relabeling $\alpha \to \alpha \times d\beta$, where $\beta(g)$ is a 1-chain, means that the phases α are classified by $H^2(G, U(1))$. This is **discrete torsion.** Unlike the anomaly which is an intrinsic part of the definition of how our symmetry lines interact, the discrete torsion is part of the data of gauging.

An anomaly prevents from sensibly gauging, since it means that “adjacent” configurations in the sum over networks cancel against each other, giving zero all together.

Partial traces

To compute any correlation function in the orbifold (i.e. gauged) theory, we need to sum over all lifts to the parent theory, where a lift involves all possible networks of G lines.

As an example, the torus partition function lifts to a sum over possible topological insertions of lines along each of our cycles. Picking some consistent way of doing that, we can define partial traces by something like

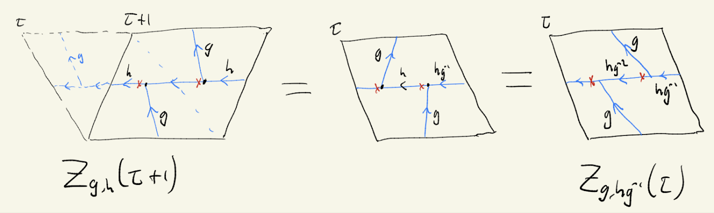


The orbifold partition function is then

$$Z = \frac{1}{|G|} \sum_{gh=hg} \frac{\alpha(g,h)}{\alpha(h,g)} Z_{g,h}.$$

Modular transformations

We can work out how these partial traces behave under modular transformations, in particular:



We would also have

$$Z_{g,h}(-1/\tau) = (\cdots)\, Z_{h,g^{-1}}(\tau).$$

Level matching

If $g^n = 1$, we have

$$Z_{g,h}(\tau + n) = \left(\prod_{j=1}^{n} \omega(g^{-1}, g^{j-1} h^{-1}, g) \right) Z_{g,h}(\tau).$$

This is how “level matching” is realized here.

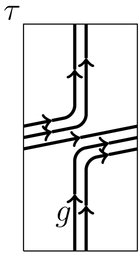
Cyclic orbifolds

Consider the case $G = \mathbb{Z}_N$. When there's no anomaly, all of the partial traces can be obtained from the untwisted sector partial traces, $Z_{0,n} = \mathrm{Tr}(g^n q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24})$. To compute $Z_{m,n}$, let $r = \gcd(m, n)$, find a and b satisfying $an + bm = r$, and then do a modular transformation with $\begin{pmatrix} a & b \\ -m/r & n/r \end{pmatrix} \in SL(2, \mathbb{Z})$.

However this computation might not be unambiguous if there is an anomaly in $H^3(\mathbb{Z}_N, U(1)) = \mathbb{Z}_N$; the indices might not be N-periodic.
Maybe it's just telling us that the group should be viewed as something bigger?

$\mathbb{Z}$ orbifold

Indeed, if we enlarge all the way up to $\mathbb{Z}$, we have the facts that $H^3(\mathbb{Z}, U(1)) = H^2(\mathbb{Z}, U(1)) = 1$, and every $Z_{m,n}$ partial trace can be built unambiguously from untwisted sector traces. For example:



If the $\mathbb{Z}_N$ is anomalous, we can actually get away with just enlarging to $\mathbb{Z}_{kN}$ (where k is the order of the anomaly class in $H^3(\mathbb{Z}_N, U(1))$).

We get a consistent orbifold!

It's not new, though.
It always turns out to be equivalent to an orbifold by a non-anomalous subgroup.

Decomposition

We realized that this connected on to a long and much more general story of decomposition. Here we imagine that we have a short exact sequence of group homomorphisms,

$$1 \longrightarrow K \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1.$$

The idea is that we are extending some effectively acting symmetry group G by some "trivially acting" additional symmetries K. If there was an anomaly ω_G, then we assume that the anomaly in the extension is the pullback $\omega_\Gamma = \pi^* \omega_G$. It turns out that we can always construct such an extension that trivializes the anomaly, $\omega_\Gamma = 1 \in H^3(\Gamma, U(1))$. For instance in our example above, the extension was

$$1 \longrightarrow \mathbb{Z}_k \longrightarrow \mathbb{Z}_{kN} \longrightarrow \mathbb{Z}_N \longrightarrow 1.$$

Basic game

Suppose first that G was non-anomalous, so Γ is as well. To gauge Γ we need to pick discrete torsion $\alpha \in H^2(\Gamma, U(1))$. For the moment we will also make the assumption that the Γ partial traces are related to the G partial traces in the most obvious way,

$$Z^{\Gamma}_{\gamma_1,\gamma_2} = Z^{G}_{\pi(\gamma_1),\pi(\gamma_2)}.$$

Then the orbifold partition function is

$$\begin{aligned} Z^{\Gamma} &= \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} \epsilon(\gamma_1,\gamma_2) Z^{\Gamma}_{\gamma_1,\gamma_2} \\ &= \frac{1}{|G|} \sum_{g,h} \left(\frac{1}{|K|} \sum_{\substack{\gamma_1\gamma_2=\gamma_2\gamma_1 \\ \pi(\gamma_i)=g_i}} \epsilon(\gamma_1,\gamma_2) \right) Z^{G}_{g_1,g_2}. \end{aligned}$$

General formula

Let $\widehat{K}_\alpha$ be the set of isomorphism classes of $\alpha|_K$-projective irreps of K.

We can define a G action on this set and divide it into G-orbits labeled by a. Here $[\rho_a]$ is some representative isomorphism class of the orbit. Let $G_a = \mathrm{Stab}([\rho_a]) \subseteq G$.
Then

$$[X/\Gamma]_\alpha = \amalg_a [X/G_a]_{\alpha_a},$$

where we can also give a formula for α_a in each term.

Some examples

- $$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1\,.$$
  $$[X/\mathbb{Z}_2 \times \mathbb{Z}_2] = 2\,[X/\mathbb{Z}_2]\,, \qquad [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_\alpha = [X]\,.$$
- $$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1\,.$$
  $$[X/D_4] = \left[X/\mathbb{Z}_2^2\right] \amalg \left[X/\mathbb{Z}_2^2\right]_\alpha\,, \qquad [X/D_4]_\alpha = [X/\mathbb{Z}_2]\,.$$
- $$1 \to \mathbb{Z}_2^2 \to S_4 \to S_3 \to 1\,.$$
  $$[X/S_4] = [X] \amalg [X/S_3]\,, \qquad [X/S_4]_\alpha = [X/S_3]\,.$$

Now with anomaly resolution

Now suppose that $\omega \in H^3(G, U(1)) \neq 1$, but $\pi^* \omega = 1 \in H^3(\Gamma, U(1))$.

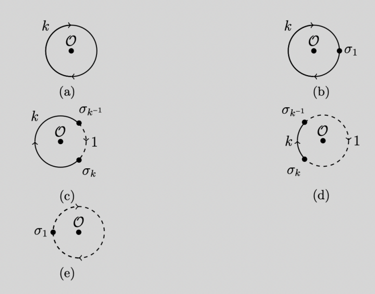
In this case, by looking at actual examples, we learn that it is not consistent to take $Z^{\Gamma}_{\gamma_1,\gamma_2} = Z^{G}_{\pi(\gamma_1),\pi(\gamma_2)}$. Instead there will be relative phases between Γ partial traces,

$$Z^{\Gamma}_{\gamma_1,\gamma_2} = (\text{phase}) \times Z^{\Gamma}_{\gamma_1 k_1, \gamma_2 k_2}.$$

To understand these phases, we need to examine our topological operators again.

Trivial symmetries

We would say that a symmetry k acts trivially if it leaves all local operators alone, $k \cdot \mathcal{O}(z, \bar{z}) = \mathcal{O}(z, \bar{z})$ for all $\mathcal{O}(z, \bar{z})$. In this case, one can argue that the line can end on a topological point operator (i.e. $V_k \neq 0$):



In general the new rule becomes that $\dim(V_{\gamma_1,\cdots,\gamma_n}) = 1$ iff $\gamma_1 \cdots \gamma_n \in K$.

Projectors

Once you gauge Γ, these operators (indeed all operators in $\mathcal{H}_\gamma$, for $\gamma \in \Gamma$) can become a true local operator, since the emanating line can be absorbed by the network of gauge lines. A bit more carefully, the local operators are associated to conjugacy classes of K.

So we get a collection of local topological point operators, whose fusion is determined by the multiplication in K. It turns out these can always be combined into projection operators Π_i obeying the fusion algebra

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i .$$

Any correlation with distinct projectors inserted will vanish! Fusing with projectors can be used to decompose the space of local operators into a direct sum.

$(d-1)$-form symmetry is decomposition

Topological point operators in 2D are what we called a 1-form symmetry.

In d dimensional QFT, the presence of a $(d-1)$-form symmetry works the same way; these local topological point operators can always be put together into combinations that are projectors which leads to decomposition of the theory.

In our case the 1-form symmetry arose when we gauged some trivially acting symmetries. In higher d one could run the same arguments with "trivially acting $(d-2)$-form symmetries".

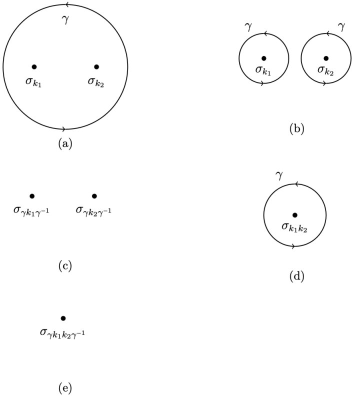
Mixed anomalies

In general in our 2D situation, we have a system of topological points and lines. Points can be either local operators, or bound to junctions. How do they interact?

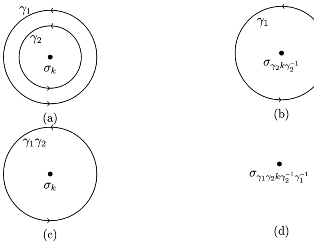
There can be mixed anomalies. If we wrap a γ line around a σ_k point operator, we could pick up a phase,

$$\gamma \cdot \sigma_k = \sigma_{\gamma k \gamma^{-1}} B(\gamma, \gamma k \gamma^{-1}).$$

The phases $B(\gamma, k)$ must satisfy certain consistency conditions. One can argue that the γ dependence should only depend on $\pi(\gamma)$.



$B(g, k)$ is a homomorphism in k.



$B(g, k)$ is a “crossed homomorphism in g.

It turns out that $B \in H^1(G, H^1(K, U(1)))$. There is a known map $d_2 : H^1(G, H^1(K, U(1))) \to H^3(G, U(1))$. For anomaly resolution we should pick B so that $d_2 B = \omega$.

Then the phases between Γ partial traces are

$$Z_{k_1\gamma_1,k_2\gamma_2} = \frac{g_1^{-1} \cdot B(\pi(\gamma_1), k_2)}{g_2^{-1} \cdot B(\pi(\gamma_2), k_1)} Z_{\gamma_1,\gamma_2},$$

We've checked that this works in many examples, but are still trying to iron out a couple of details. The paper should appear soon!

Thanks!!