Adiabatic continuity, anomaly preserving compactifications, and calculable confinement

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arXiv:2201.06166, 2205.11339

Adiabatic continuity

Adiabatic continuity is the idea that in (non-)supersymmetric gauge theories, Yang-Mills, QCD, etc non-perturbative strong coupling phenomena - e.g. confinement, chiral symmetry breaking, mass gap, multi-branched vacuum structure can be continuously connected to arbitrarily weak coupling regimes!

All of these phenomena were believed to take place necessarily at strong coupling (since mid-70s). This belief is still common in contemporary literature, but not true.

Adiabatic continuity

In the last 15 years, it is understood that adiabatic continuity can be achieved by judiciously chosen circle compactifications on R3 x S1, matter content, boundary conditions. These are all 1-loop quantum effects stabilizing 0-form part of center (whenever it exists). But R3 x S1 is not the subject of my talk today.

(MU, Yaffe, Shifman, Poppitz, Argyres, Cherman, Schaefer, Sulejmanpasic, Anber, Dunne, Misumi, Susy: Hollowood, Khoze, Dorey, …….)

Rather, I will introduce another realization of adiabatic continuity, which provides new complementary insights, and some intriguing puzzles.

Introduce a more robust mechanism of adiabatic continuity, which uses classical 't Hooft flux.

Two challenging questions

⇒Can we continuously connect 4d physics (in generic non-susy theories) with 2d physics? (without intervening phase transitions?)

⇒Technical, but equally important: How do we formulate semi-classics in 't Hooft flux background in a thermodynamic limit?

Adiabatic continuity on $R_2 \times T_2$?

- At large $T_2 x R_2$, SU(N) gauge theory, SU(N) with the insertion of 't Hooft flux, and PSU(N) theory possess identical local dynamics. Local correlation functions (e.g mass gap) are the same.

- But at small $T_2 \times R_2$, these theories are **not** locally same due to subtle quantum effects! In fact, studying the SU(N) with the insertion of 't Hooft flux is far more useful than standard periodic b.c. We will show this in detail.

Locally same, globally different theories!

SU(N) and PSU(N) theories are locally the same on large M4. The latter is the marriage of QFT with TQFT. It cannot differ locally.
But bundle topologies are different, e.g.

$$Z_{SU(N)} = \sum_{W \in \mathbb{Z}} e^{\mathrm{i}\theta W} Z_W$$

$$Z_{SU(N)}(\ell, m) = \sum_{W \in \mathbb{Z}} e^{\mathrm{i}\theta \left(W + \frac{(\ell \cdot m)}{N}\right)} Z_W(\ell, m)$$

$$Z_{PSU(N)_p} = \sum_{\substack{W \in \mathbb{Z} \\ \ell, m \in (\mathbb{Z}_N)^3}} e^{\mathrm{i} \frac{2\pi}{N} p \, (\ell \cdot m)} e^{\mathrm{i}\theta \left(W + \frac{(\ell \cdot m)}{N}\right)} Z_W(\ell, m)$$

$$S = \frac{8\pi^2}{g^2} \left| W + \frac{(\ell \cdot m)}{N} \right| \in \frac{S_I}{N} \mathbb{Z}^{\geq 0}$$

Action of BPS saddles in PSU(N)
What is the implication for SU(N) ?

The latter partition functions can be defined provided $H^2(M_4, \mathbb{Z}_N)$ nontrivial.

•The 't Hooft flux background is commonly used to provide kinematical constraints. Seiberg et.al. used it to demonstrate mixed anomalies involving 1-form symmetries and constraints on possible IR-phases. We use it to study semi-classical dynamics.

This talk:

⇒Yang-Mills theory : Multi-branch structure, string tensions

⇒QCD with fundamental fermions. (Derivation of chiral Lagrangian at small $T_2 \times R_2$ and matching to large $T_2 \times R_2$)

⇒N=1 SYM

⇒QCD with (S/AS) rep. fermions (S/AS), chiral theories

Yang-Mills on $R_2 \times T_2$

- Take symmetric T_2, size smaller than strong scale. Compactified directions x_3, x_4.

- Classical minima given by flat connections, $F_{34}=0$. Let us name holonomies in the compact directions P_3, P_4. Classically, each holonomy takes values in the maximal torus T_N.

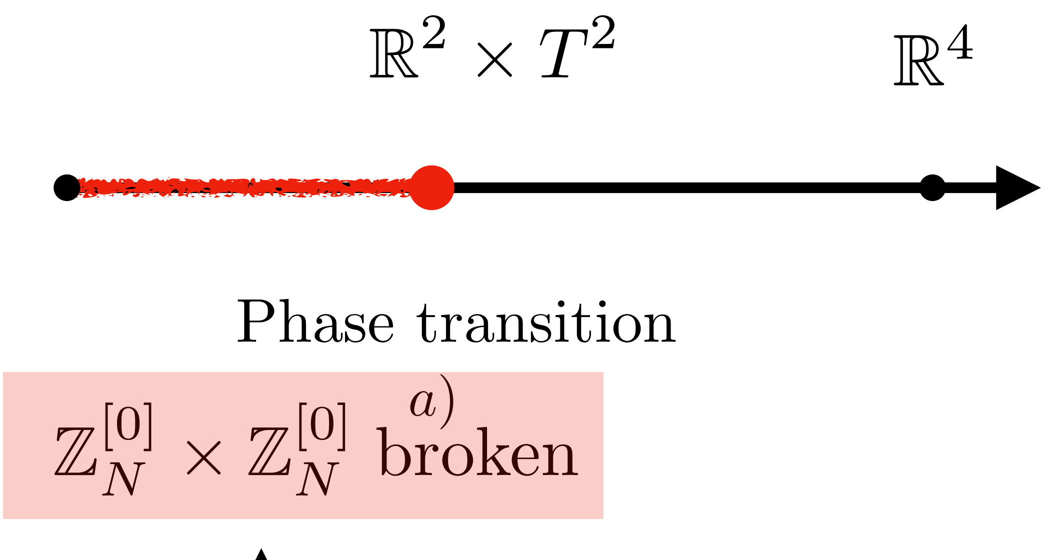
$$P_3 = \mathcal{P}\exp\left(\mathrm{i}\int_0^L a_3 \mathrm{d}x_3\right), \quad P_3 = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1}, \mathrm{e}^{\mathrm{i}\alpha_2}, \ldots, \mathrm{e}^{\mathrm{i}\alpha_N}),$$

- Classical moduli space: $\mathcal{M}_{\mathrm{cl}} = (\mathbf{T}_N)^2/S_N$.

- Q: What happens to the moduli space quantum mechanically? What happens in the presence of 't Hooft flux classically and quantum mechanically?

Yang-Mills on R2 x T2 without and with ’t Hooft flux

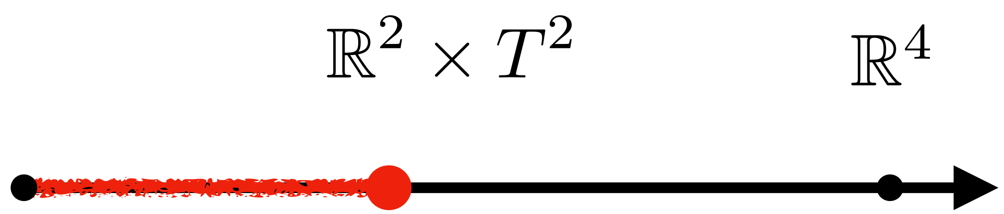
1-form symmetry: $\left(\mathbb{Z}_N^{[1]}\right)_{4\mathrm{d}} \xrightarrow{T^2 \text{ compact.}} \left(\mathbb{Z}_N^{[1]}\right)_{2\mathrm{d}} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}.$



$$V_{1-\mathrm{loop}} = -\frac{2}{\pi^2 L^4} \sum_{(n_3,n_4)\in\mathbb{Z}^2\setminus\mathbf{0}} \frac{|\mathrm{tr}(P_3^{n_3} P_4^{n_4})|^2}{(n_3^2+n_4^2)^2}$$

Yang-Mills on R2 x T2 without and with 't Hooft flux

1-form symmetry: $\left(\mathbb{Z}_N^{[1]}\right)_{4\text{d}} \xrightarrow{T^2 \text{ compact.}} \left(\mathbb{Z}_N^{[1]}\right)_{2\text{d}} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}.$



Phase transition

a)

$\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ broken

$$V_{1-\text{loop}} = -\frac{2}{\pi^2 L^4} \sum_{(n_3,n_4)\in\mathbb{Z}^2\setminus\mathbf{0}} \frac{|\text{tr}(P_3^{n_3} P_4^{n_4})|^2}{(n_3^2+n_4^2)^2}$$

$\mathbb{R}^2 \times T^2$ with 't Hooft flux

Adiabatic continuity

b)

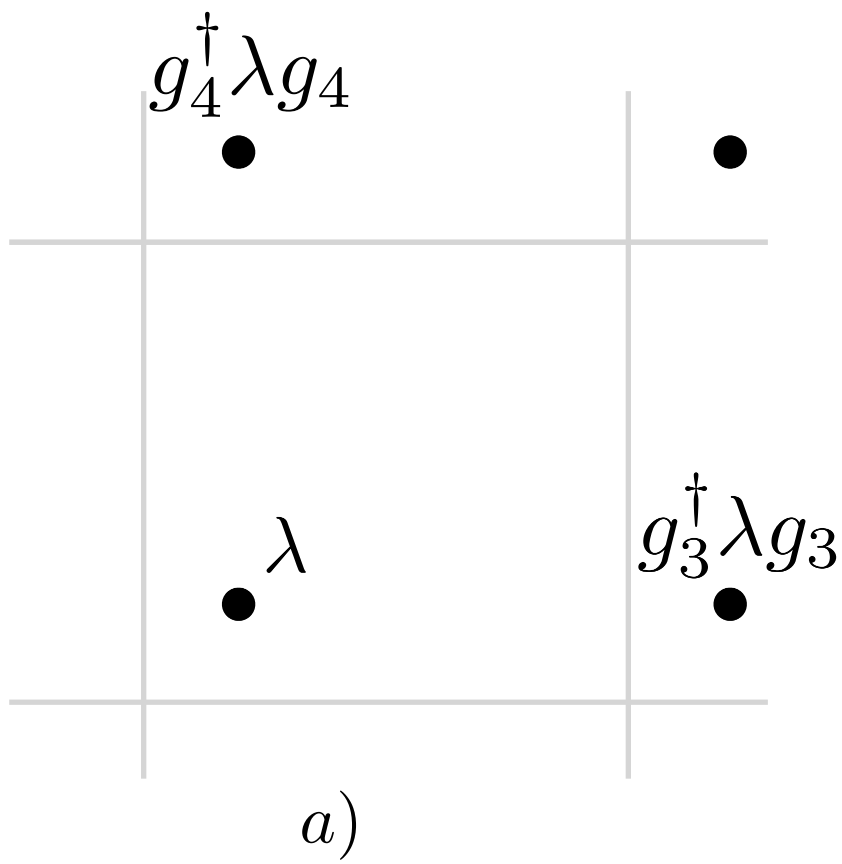
$\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ unbroken

- Opposite to periodic case
- Why does it occur?

Activating a fixed 't Hooft flux= Activating a fixed $B^{(2)}$

't Hooft 78

YM, SYM,
QCD(adj),
N=2 SYM



λ : adjoint matter

$$g_3(L_4)^\dagger g_4(0)^\dagger = g_4(L_3)^\dagger g_3(0)^\dagger \exp\left(\frac{2\pi \mathrm{i}}{N} n\right).$$

Co-cycle or consistency condition

Classical minima is again in terms of flat connections, and the Polyakov loops are dictated by transition matrices. With flux, classical minima is given by non-commuting clock and shift matrices.

$$P_3 = g_3\, \mathcal{P}\mathrm{e}^{\mathrm{i}\int_0^L a_3 \mathrm{d}x_3} = S, \qquad P_4 = g_4\, \mathcal{P}\mathrm{e}^{\mathrm{i}\int_0^L a_4 \mathrm{d}x_4} = C.$$

$$C \propto \mathrm{diag}(1, \omega, \ldots, \omega^{\mathrm{N}-1}), \qquad (S)_{i,j} \propto \delta_{i+1,j} \text{ with } \omega = e^{2\pi i/N}$$

Think of these as two non-commuting adjoint Higgs field. Hence

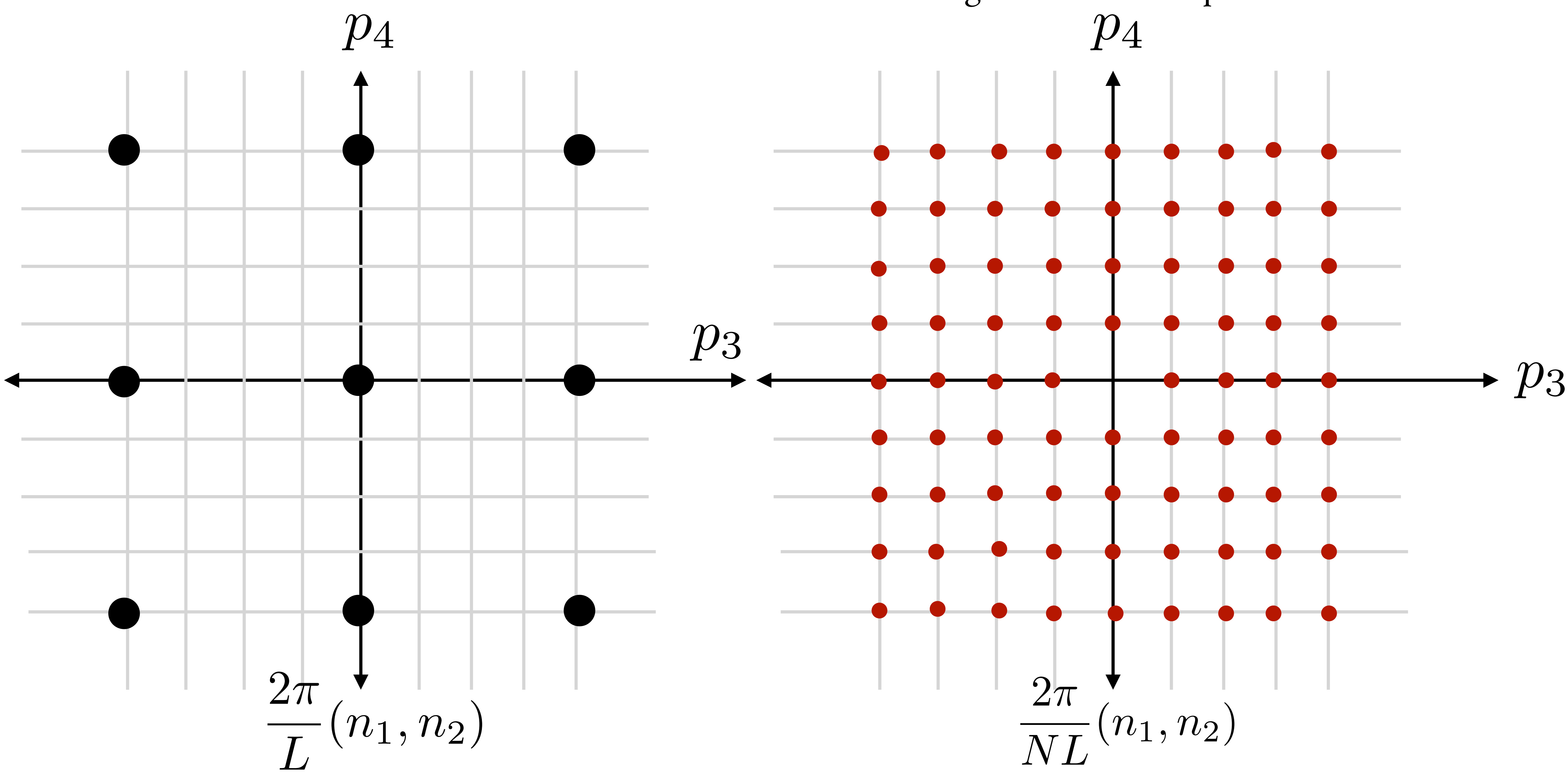
$$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N$$

Perturbative spectrum on R2 x T2

without flux

with flux

No gapless modes in 2d!
Forms a continuum in the N ➡∞!
Large-N volume independence. Comment!



Wilson loops $\Longrightarrow$ Perimeter law in perturbation theory. (TQFT in pert. th.)
How about non-perturbatively? Is this TQFT destabilized?

Fractional instantons $\Longrightarrow$ Center vortices

$$Q_{\rm top} = \frac{1}{8\pi^2}\int_{T^4} {\rm tr}\left((\tilde{F} - B)^2\right) \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma}n_{\mu\nu}n_{\rho\sigma}}{8} + \mathbb{Z}.$$

('t Hooft, van Baal,...)

$$\mathrm{Re}(S_{\rm YM}) \geq \frac{8\pi^2}{g^2}|Q_{\rm top}| = \frac{8\pi^2}{Ng^2},$$

Assuming solution that satisfies BPS-bound exists.

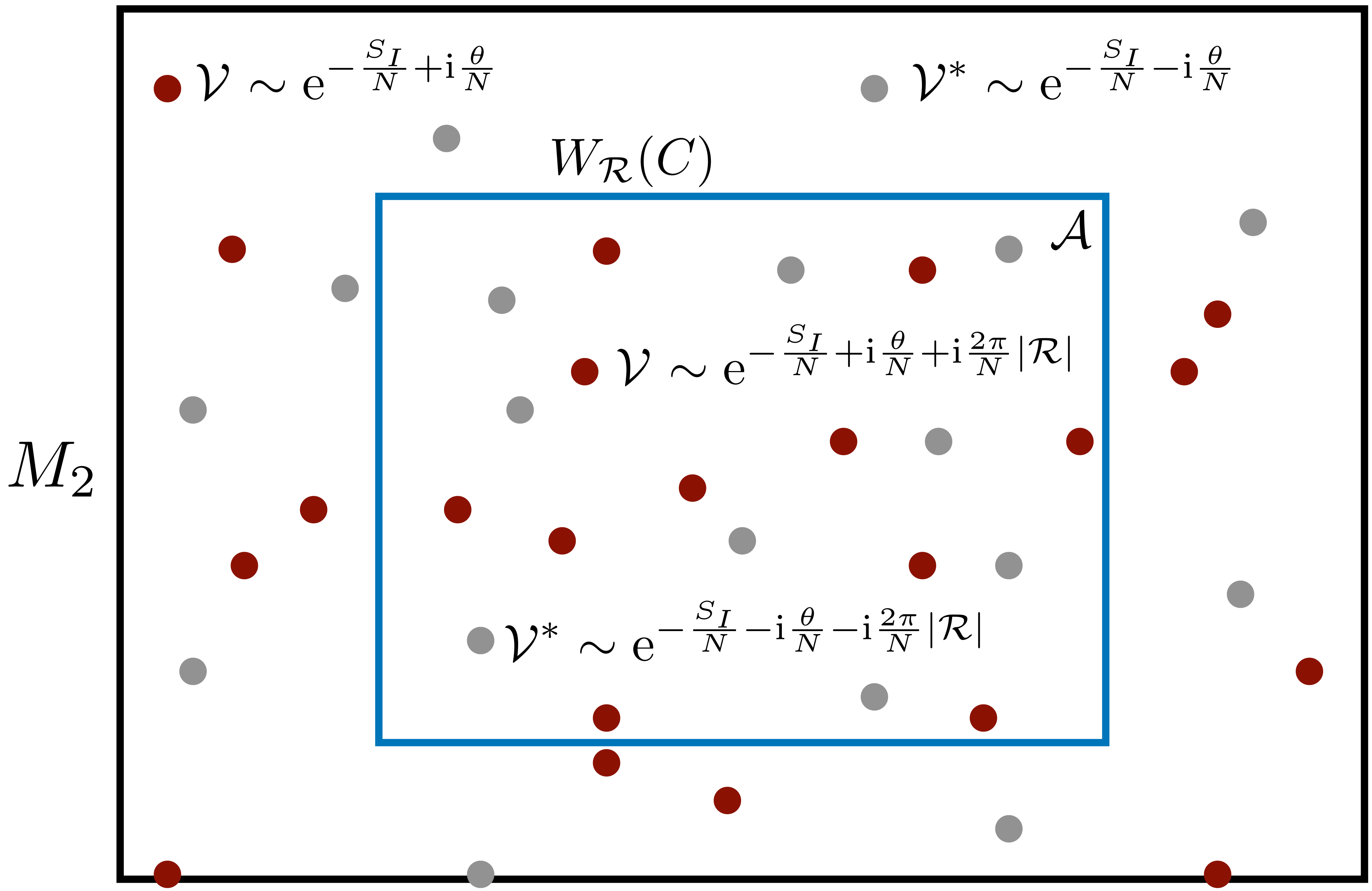
There is compelling evidence from lattice that there is. (Gonzalez-Arroyo, Montero, Garcia-Perez 1990s). Furthermore,

$$W_{\mathcal{R}}(C) = \exp(2\pi i|\mathcal{R}|/N)$$

when a vortex is inside the loop. Non-trivial mutual statistics between the Wilson loop and the vortex!

Proliferation of vortices and semi-classics

Semi-classical description here is very similar to charge-N abelian Higgs model in 2d.



First, ignore the Wilson loop. Let us just look to partition function.

Proliferation of vortices and semi-classics

$$Z(\theta) = \sum_{n,\overline{n}\geq 0} \frac{V^{n+\overline{n}}}{n!\overline{n}!} K^{n+\overline{n}} \mathrm{e}^{-(n+\overline{n})S_I/N} \mathrm{e}^{\mathrm{i}(n-\overline{n})\theta/N} \delta_{n-\overline{n}\in N\mathbb{Z}},$$

Integer top. charge

$$Z(\theta) = \sum_{n,\overline{n}\geq 0} \frac{V^{n+\overline{n}}}{n!\overline{n}!} K^{n+\overline{n}} \mathrm{e}^{-(n+\overline{n})S_I/N} \mathrm{e}^{\mathrm{i}(n-\overline{n})\theta/N} \sum_{k=0}^{N-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}\, k}{N}(n-\overline{n})}$$

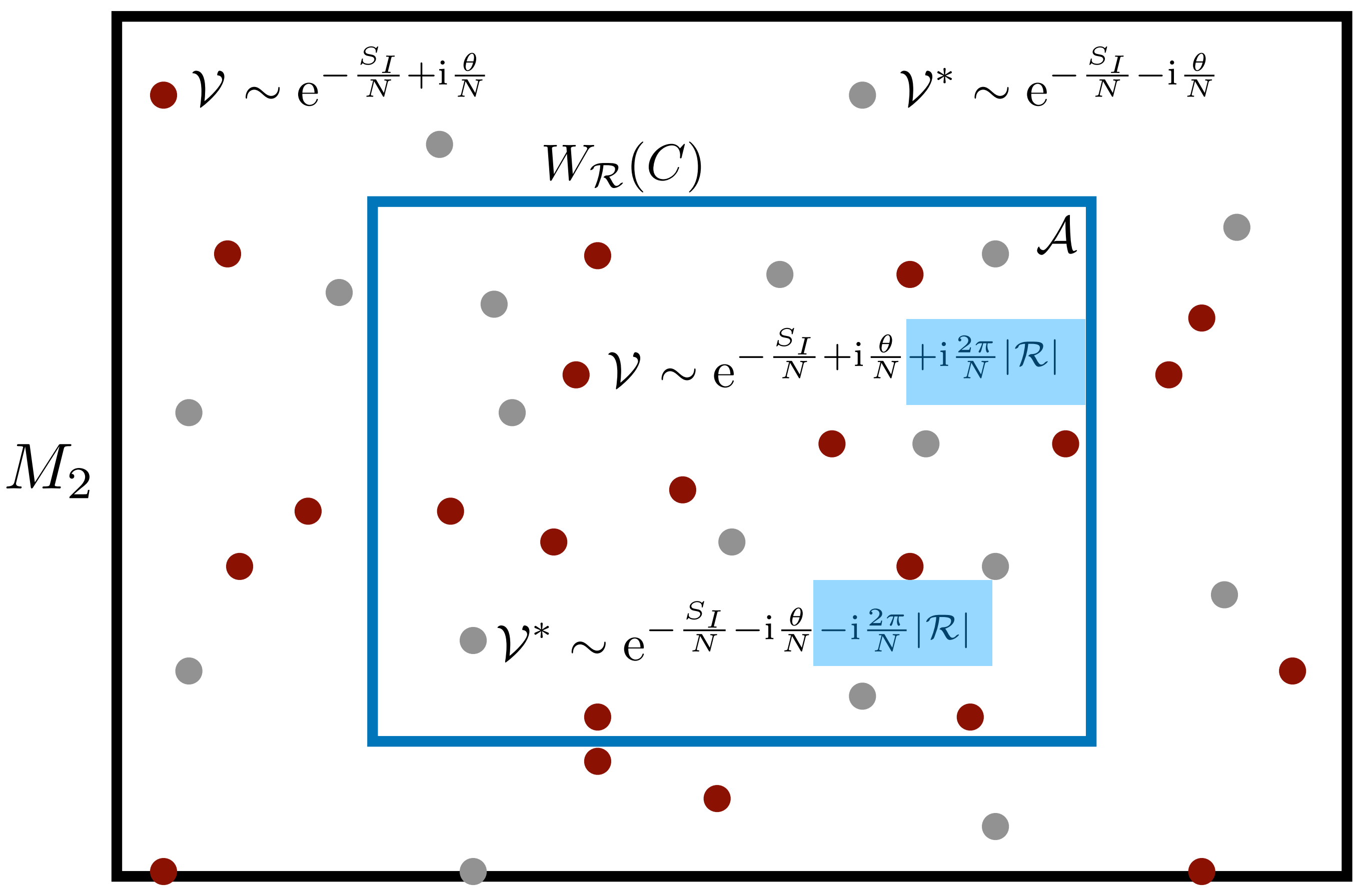
$$= \sum_{k=0}^{N-1} \exp\left[VK\mathrm{e}^{-S_\mathrm{I}/N+\mathrm{i}(\theta-2\pi k)/N} + VK\mathrm{e}^{-S_\mathrm{I}/N-\mathrm{i}(\theta-2\pi k)/N} \right]$$

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(-2K\mathrm{e}^{-S_I/N} \cos\left(\frac{\theta-2\pi k}{N} \right) \right) \right],$$

Multi-branched vacuum structure

$$E_k(\theta) = -\Lambda^2 (\Lambda L_s)^{5/3} \cos\left(\frac{\theta - 2\pi k}{N} \right).$$

$$\langle W_{\mathcal{R}}(C)\rangle = \frac{1}{Z(\theta)} \sum_{n_1,n_2,\overline{n}_1,\overline{n}_2} \frac{\mathcal{A}^{n_1+\overline{n}_1}(V-\mathcal{A})^{n_2+\overline{n}_2}}{n_1!\,n_2!\,\overline{n}_1!\,\overline{n}_2!} \left(K\,\mathrm{e}^{-S_I/N}\right)^{n_1+n_2+\overline{n}_1+\overline{n_2}}$$
$$\times\, \mathrm{e}^{\mathrm{i}(n_1+n_2-\overline{n}_1-\overline{n}_2)\theta/N}\, \mathrm{e}^{2\pi\mathrm{i}(n_1-\overline{n}_1)|\mathcal{R}|/N}\, \delta_{n_1+n_2-\overline{n}_1-\overline{n}_2\in N\mathbb{Z}}$$
$$= \frac{1}{Z(\theta)} \sum_{k=0}^{N-1} \mathrm{e}^{-VE_k(\theta)} \exp\Big(-\mathcal{A}(E_k(\theta+2\pi|\mathcal{R}|)-E_k(\theta))\Big).$$



Extra phases: Due to non-trivial Mutual statistics of Wilson loops with center vortex.

Only the ones inside the loop acquire these phases.

$$T_{\mathcal{R}}(\theta) = E_0(\theta+2\pi|\mathcal{R}|) - E_0(\theta)$$
$$\sim \Lambda^2(\Lambda L_s)^{5/3}\left(\cos\frac{\theta}{N} - \cos\frac{\theta+2\pi|\mathcal{R}|}{N}\right).$$

In the V →∞ limit, we obtain Finite string tension.

Formally

To all orders in pert. theory, long distance is described by a $\mathbb{Z}_N$ TQFT.

This TQFT in 2d is not robust! And it is destabilized by local topological operators. (Cherman, Jacobson, Neuzil 2021). These are semi-classic center-vortices!

$$S = \frac{iN}{2\pi}\int_M \varphi\, da - \zeta \int_M d\mu\, \cos(\varphi + \theta/N)$$

IR theory is a deformed TQFT. (Nguyen, MU, Tanizaki, Sulejmanpasic 2023)

Usually, gapped phases of matter are classified according to TQFTs and YM is presented as trivially gapped phase. Our construction gives a more refined data. IR of YM is a dTQFT on R2 x T2. Multi-branch structure is a remnant of dTQFT.

Why is this compactification special?

Mixed anomaly: $\mathbb{Z}_N^{[1]}$ and CP at $\theta = \pi$.

$$Z_{\theta+2\pi}[B_{\rm 4d}] = \exp\left(\frac{{\rm i}N}{4\pi}\int_{M_2\times T^2} B_{\rm 4d}\wedge B_{\rm 4d}\right) Z_\theta[B_{\rm 4d}].$$

Gaiotto, Kapustin, Komargodski, Seiberg, 2017

$$CP : Z_{\theta=\pi}[B_{\rm 4d}] \to Z_{\theta=-\pi}[B_{\rm 4d}]$$
$$= \exp\left(-\frac{{\rm i}N}{4\pi}\int_{M_2\times T^2} B_{\rm 4d}\wedge B_{\rm 4d}\right) Z_{\theta=\pi}[B_{\rm 4d}].$$

Vacuum at **θ**= **π** can not be trivial.

Compactification and mixed anomaly

$$B_{4\mathrm{d}} = B_{2\mathrm{d}} + A_3 \wedge \frac{\mathrm{d}x_3}{L_s} + A_4 \wedge \frac{\mathrm{d}x_4}{L_s} + \frac{2\pi n_{34}}{N} \frac{\mathrm{d}x_3 \wedge \mathrm{d}x_4}{L_s^2}.$$

- $B_{2\mathrm{d}}$: 2-form gauge field for $\mathbb{Z}_N^{[1]}$, which couples to $W_{\mathcal{R}}(C)$ inside M_2.
- A_3: 1-form gauge field for one of $\mathbb{Z}_N^{[0]}$, which couples to P_3.
- A_4: 1-form gauge field for another $\mathbb{Z}_N^{[0]}$, which couples to P_4.

$$Z_{\theta+2\pi}^{(n_{34})}[B_{2\mathrm{d}}, A_3, A_4] = \exp\left(\mathrm{i}\, n_{34} \int_{M_2} B_{2\mathrm{d}} - \frac{\mathrm{i}N}{2\pi} \int_{M_2} A_3 \wedge A_4 \right) Z_{\theta}^{(n_{34})}[B_{2\mathrm{d}}, A_3, A_4].$$

This is why it is special.

Mixed anomaly between 1-form and CP survives when n(34) is non-zero.
$\theta = \pi$ can not be trivial. Consistent with semi-classical description.

Can we do similar analysis in QCD with fundamental quarks?

According to 't Hooft (81), the answer is no. With the introduction of fundamental matter, we loose $\mathbb{Z}_N$ one-form symmetry and it is not possible to impose 't Hooft's tbc consistently.

't Hooft (81): "Note that these classes disappear if a field in the fundamental representation of SU(N) is added to the system (these fields would make unacceptable jumps at the boundary)."

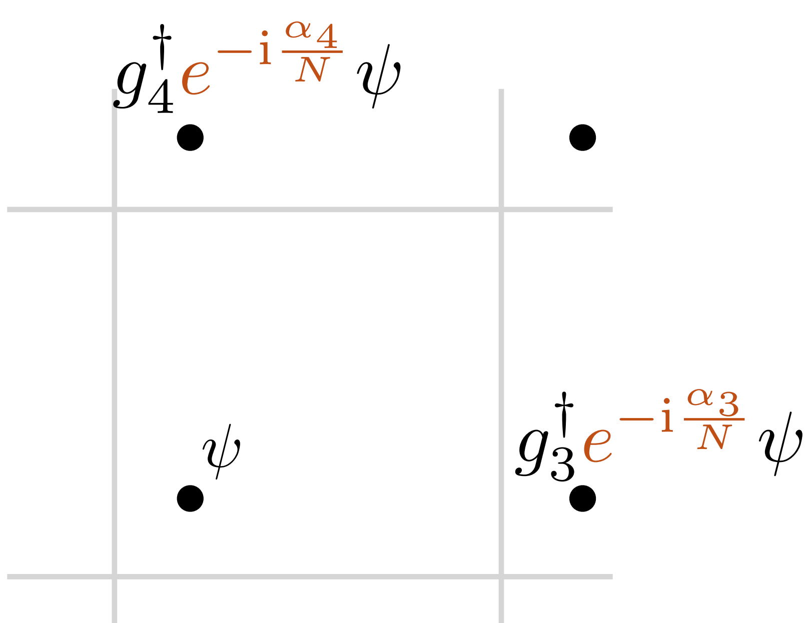
This actually turns out to be too fast, and there are more than one way around it.

’t Hooft flux in the presence of fundamental quarks

Obstacle: One cannot naively introduce ’t Hooft flux in the presence of fundamental matter field. (No center symmetry).

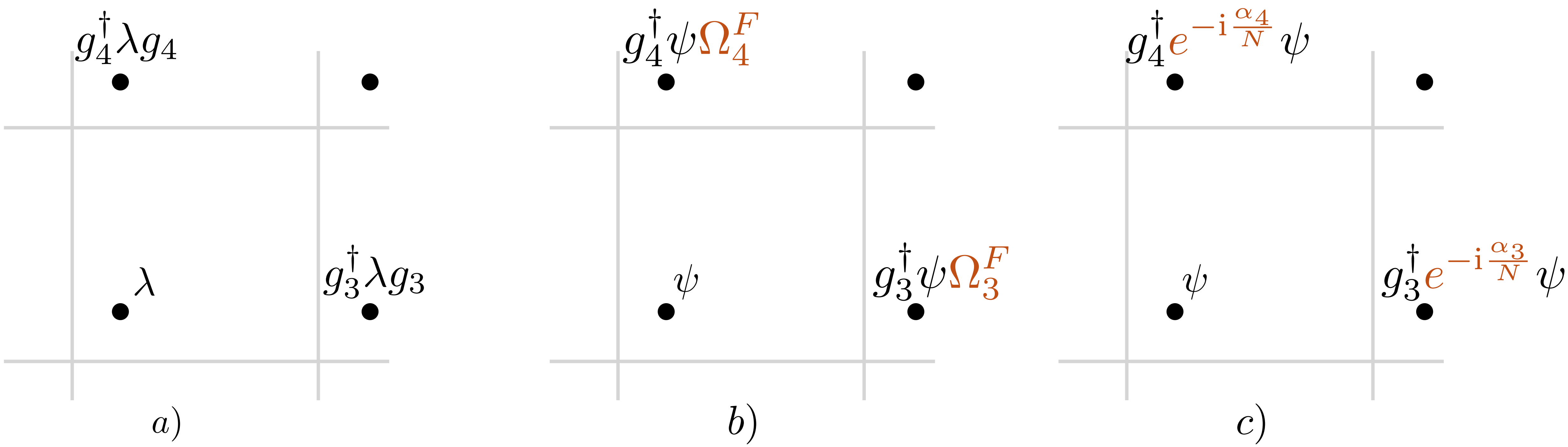
Way around: Turn on a $U(1)_B$ baryon magnetic flux background.
Since $U(1)_B = U(1)_q / \mathbb{Z}_N$, ’t Hooft flux can still be inserted through the common center group of SU(N) and $U(1)_B$

Misumi, Sakai, Tanizaki 2017



$$\psi(x_3 + L, x_4) = e^{-i\alpha_3(x_4)/N} g_3(x_4)^\dagger \psi(x_3, x_4),$$
$$\psi(x_3, x_4 + L) = e^{-i\alpha_4(x_4)/N} g_4(x_3)^\dagger \psi(x_3, x_4)$$

Two ways around:



YM, SYM,
QCD(adj),
N=2 SYM

$N_f = N_c$ QCD(F)

N_f-flavor QCD(F)

Turning on 't Hooft flux in QCD(F) possible by activating a flavor symmetry twisted b.c. or by activating $U(1)_B = U(1)_q / \mathbb{Z}_N$

The perturbative massless spectrum in baryon number background

$\Longrightarrow$ Solve Dirac eq. with t.b.c.

$\Longrightarrow$ Nf 2d massless fermions

$\Longrightarrow$ Zero modes: Jacobi Theta function, quasi-periodic.

= twisted boundary conditions satisfied.

Non-abelian Bosonization

2d N_f-flavor massless Dirac fermions can be mapped to 2d level-1 $U(N_f)$ WZW

$$S=\frac{1}{8\pi}\int_{M_2}\mathrm{tr_f}(\mathrm{d}\tilde{U}\wedge\star\mathrm{d}\tilde{U}^\dagger)+\frac{1}{12\pi}\int_{M_3}\mathrm{tr_f}[(\tilde{U}^\dagger\mathrm{d}\tilde{U})^3],$$

From Non-abelian Bosonization, Witten

Almost chiral Lagrangian.... Wait a bit more. Center-vortex generate

$$\Delta S\sim-\frac{1}{L^2}\mathrm{e}^{-S_\mathrm{I}/N}\left(\mathrm{e}^{\mathrm{i}\theta/N}(\det\tilde{U})^{1/N}+\mathrm{e}^{-\mathrm{i}\theta/N}(\det\tilde{U}^\dagger)^{1/N}\right).$$

which lifts one of the gapless degrees of freedom. So, IR is SU(Nf)1 level-1 WZW model with central charge Nf-1.

In the large-N limit, the center-vortex term takes the form

$$\Delta S \sim \frac{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}{N^2} \left(\mathrm{i} \ln\left(\det(\tilde{U})\right) - \theta \right)^2$$

Which gives η' mass, consistent with the Witten-Veneziano formula.

If one actually assumes that adiabatic continuity holds, this construction is a derivation of the chiral Lagrangian. Let us show this.

What happens at large T2 x R2? Chiral Lagrangian perspective

If you compactify chiral largrangian on large T2 x R2 and consider physics at length scales larger than T2 size, you land on 2d Principle Chiral Model.

$$S = \frac{1}{\lambda} \int_{M_2} \mathrm{tr}_{\mathrm{f}} (\mathrm{d}U \wedge \star \mathrm{d}U^\dagger), \qquad \frac{1}{\lambda} = L^2 f_\pi^2$$

PCM is asymptotically free and gapped in 2d. This does not look anything like what we obtained at small T2 x R2.

But since we are considering theory in U(1)$_B$ background, we must couple it to baryon (Skyrmion) current, and this changes the story.

What happens at large T2 x R2? Chiral Lagrangian perspective

U(1)$_B$ background must be coupled to baryon (Skyrmion) current:

$$J_{\mathrm{B}} = \frac{1}{24\pi^2}\mathrm{tr_f}[(U^\dagger \mathrm{d}U)^3]$$

$$\int_{M_4} A_{\mathrm{B}} \wedge J_{\mathrm{B}} = \int_{M_5} \mathrm{d}A_{\mathrm{B}} \wedge J_{\mathrm{B}}.$$

$$\int_{M_3 \times T^2} \mathrm{d}A_{\mathrm{B}} \wedge J_B = \frac{1}{12\pi}\int_{M_3} \mathrm{tr_f}[(U^\dagger \mathrm{d}U)^3] =: \Gamma_{\mathrm{WZW}}[U]$$

This theory flows to level-1 WZW CFT.
Perfect match between microscopic QCD analysis and macroscopic chiral Lagrangian analysis.

Outlook

- All of the known non-trivial strong coupling (confinement, chiral S.B., multi-branch structure etc.) phenomena can be continuously connected to weak coupling!

- All of the above are NP phenomena, controlled by $\exp[-8\pi^2/(g^2N)]$ effects that can take place both at weak and strong coupling.

- Two genuinely different confinement mechanisms in two reliable semi-classical regimes. Monopole-instantons or magnetic bions on $R_3 \times S_1$ (not discussed in this talk) vs. center vortex on $R_2 \times T_2$. A quite interesting puzzle!

- Since everything matches to strong coupling expectations, it is impossible not to speculate that the semi-classical basis (fractional instanton saddles and critical points at infinity) may actually be a complete basis in the sense of resurgence. This is a quite intriguing possibility.