Mordell-Weil Torsion in the Mirror of Multi-Sections

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Based on
- arXiv:1408.4808 with: D. Klevers, D. Mayorga, H. Piragua and J. Reuter
- arXiv:1604.00011 with: J. Reuter and T. Schimannek

Regional Meeting 2016, Blacksburg
April 23rd 2016

Outline

1. **F-theory, singular fibers and sections**
2. **Torus Fibers in 2D Ambient Spaces**
3. **Generalizing to 3D Ambient Spaces**
4. **Example: Nef** $(122, 0)$
5. **Conclusion and Outlook**

The F-theory picture

F-theory: Take the Tye IIB axio dilaton: $\tau = C_0 + i g_s^{-1}$ with

- Theory invariant under $\tau \to \frac{a\tau+b}{c\tau+d}$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$
- In addition the C_2 and B_2 must transform as a doublet:

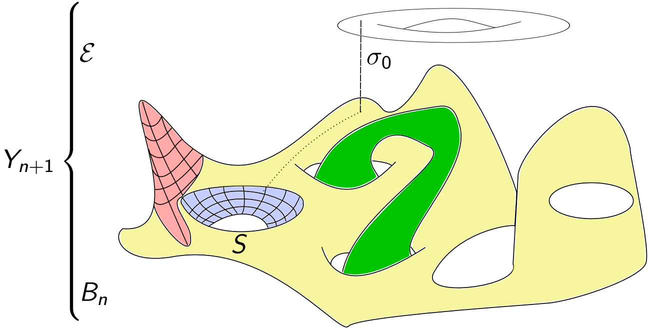
$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} aC_2 + B_2 \\ cC_2 + dB_2 \end{pmatrix} ,$$

- We interpret this structure as coming from the geometry of a torus $\mathcal{E}$
- The full geometry is a *torus*-fibered n-fold Y_n

$$\begin{array}{ccc} \mathcal{E} & \to & Y_{n+1} \\ & & \downarrow \\ & & B_n \end{array}$$

- In the M-theory dual picture, the F-theory fiber Volume is taken to zero, only the Base B_n is physical

F-theory Geometry



- The zero section σ_0 tracks the varying fiber over every point in the base B_n

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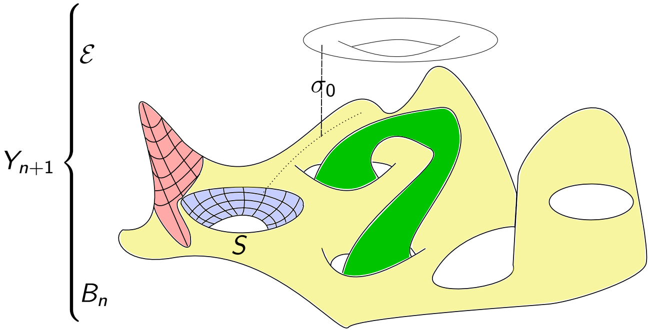

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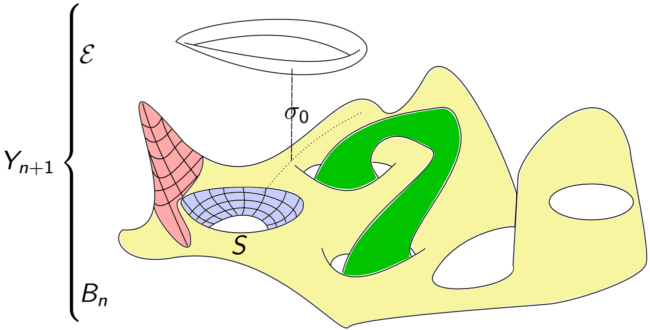

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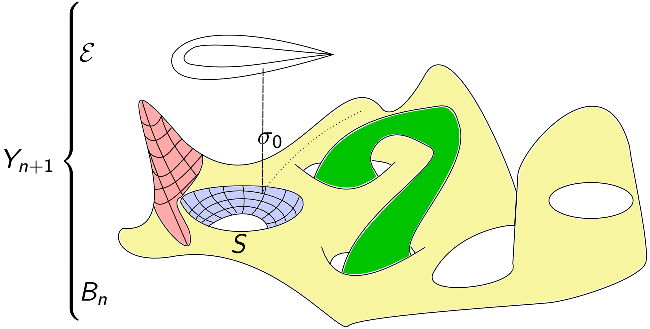
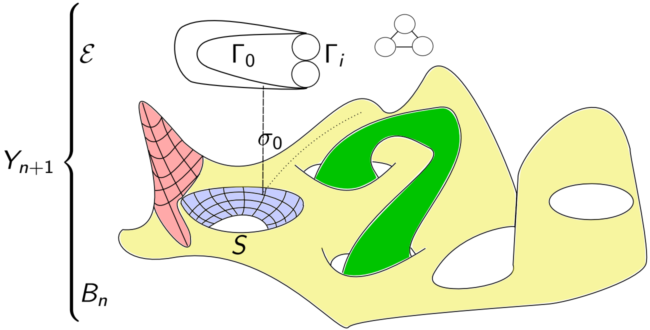


- The zero section σ_0 tracks the varying fiber over every point in the base B_n

Over certain codimension 1 (or higher) loci the fiber degenerates:

- $\tau \to \infty$: @ D7-brane divisor in the base S
- The whole CY geometry singular resolution in the fiber required

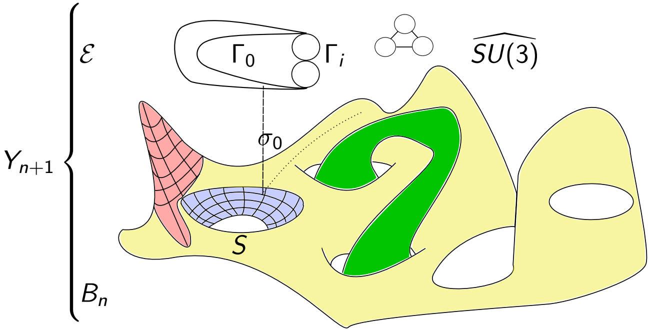
F-theory Geometry



The singularity is of ADE type and can be resolved by gluing in a tree of $\mathbb{P}^1$'s

- The $\mathbb{P}^1$'s introduce
  - cycles: Γ_i and divisors: D_i with $i = 1...n$
  - intersecting $\Gamma_i \cdot D_j = -C_{ij}$
- The zero point σ_0 identifies the affine Node Γ_0
- $C_{i,j}$ the Cartan matrix of an affine Lie-group.

F-theory Geometry



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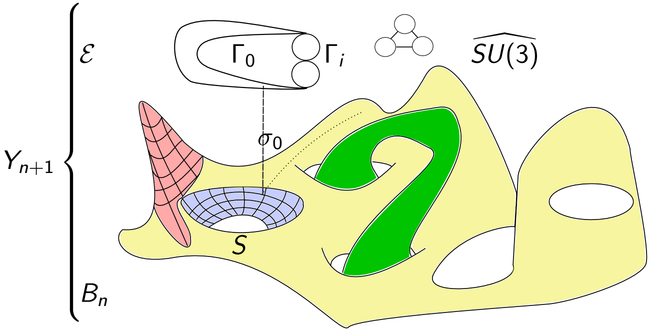


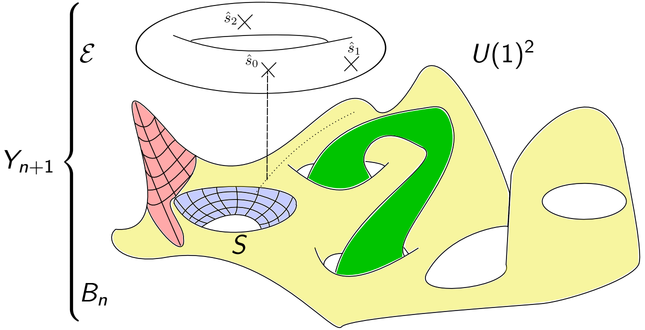
Three sources of Vector-multiplets:

- Cartan-Vector multiplets from the M-theory C_3-form reduction:

$$\int_{\Gamma^i} C_3 = A_1^i$$

- M2 branes wrap $\Gamma_i \cup \Gamma_{i+}$.. that become the massless W bosons when we take the F-theory limit.

F-theory Geometry

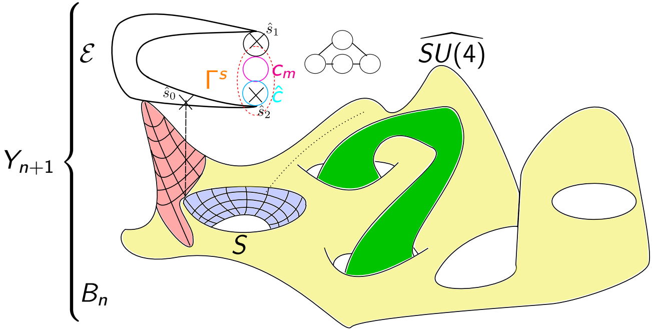


Three sources of Vector-multiplets:

- Additional rational sections s_j generate the *free Mordell-Weil group* with $s_j \cdot \mathcal{E} = 1$ give the Shioda map [Vafa Morrison, Morrison Park...]

$$\sigma_i = s_j - s_0 + \pi(K_B^{-1}) + \sum a_n D_n \,, \quad C_3 = \sigma_j \wedge A_1^j$$

F-theory Geometry



At codimension two, the fiber degenerates further

- Matter curves from codimension two splits of the curve $\mathcal{E}$:

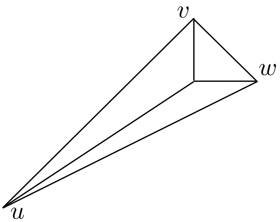
$$\Gamma^s \to c_m + \hat{c}\,, \text{ with weights: } (\lambda_i, q_i) = (D_i, \sigma_i) \cdot c_m\,.$$

Fiber VS. Base

Thanks to the splitting of fiber and base there are two general trends to classify F-theory compactifications:

1. Classification of fibers $\mathcal{E}$
   - The Kodaira classification of singular fibers [Kodaira]
   - Tates algorithm [Tate, Katz, Morrison, Schäfer-Nameki, Sully...]
   - Classification of fibers in various ambient spaces [Braun,Grimm,Keitel; Klevers, Pena, Piragua, O., Reuter]
2. Classification of bases B_n
   - Classification of two dimensional bases [Morrison, Taylor]
   - Classification of non-higgsable clusters [Morrison, Taylor; Grassi, Halverson, Shaneson, Taylor]
3. **Bonus ingredient in 4D**: Classification of fluxes [Bizet, Klemm, Lopes; A.Braun Watari]

The Fiber description



Canonical choice: Weierstrass Form

The Weierstrass form as vanishing degree six polynomial $P_{(1,3,2)}[6]$ in $[u, v, w]$:

$$v^2 - w^3 - f(b)wu^4 - g(b)u^6 = 0, \qquad \Delta = 27g^2 + 4f^3$$

- **zero section:** σ_0: $[u, v, w] = [0, 1, 1]$
- **Base Dependency** in only two sections $f(b), f(b)$
- **Discriminant:** $\Delta = 0 \rightarrow$ singularity directly visible
  $\rightarrow$ Classification of all codimension 1,2,3 ideals V_i : with $\Delta_i = 0$ possible

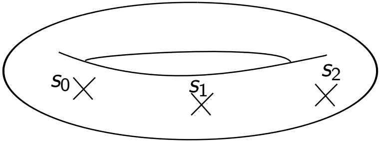
The Kodaira Classification

ord(f)	ord(g)	ord(Δ)	Fiber Type	Singularity Type	group
≥ 0	≥ 0	0	smooth	none	-
0	0	n	I_n	A_{n-1}	$SU(n)$
≥ 1	1	2	II	none	-
1	≥ 2	3	III	A_1	SU(2)
≥ 2	2	4	IV	A_2	SU(3)
2	≥ 3	$n+6$	I_n^*	D_n+4	$SO(2n+8)$
≥ 2	3	$n+6$	I_n^*	D_n+4	$SO(2n+8)$
≥ 3	4	8	IV^*	E_6	E_6
3	≥ 5	9	III^*	E_7	E_7
≥ 4	5	10	II^*	E_8	E_8
			I_1	none	$U(1)^n$?
			I_1	none	Z_{n_j} ?
			I_1	none	$1/Z_{n_j}$?

$$v^2 - w^3 - f(b)wu^4 - f(b)u^6 = 0, \qquad \Delta = 27g^2 + 4f^3$$

- What about U(1), discrete and quotient symmetries?

The Zoo of Sections



Rational Sections

Rational sections intersect the fiber once $s_i \cdot \mathcal{E} = 1$. They form the *Mordell-Weil group* und geometric addition with the zero-section as neutral element

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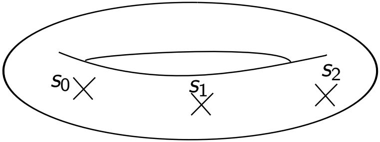

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1. **free part $\mathbb{Z}^r$:** $U(1)^r$ symmetries are obtained from the Shioda map, a vertical divisor obtained from the section [Morrison, Park]

$$\sigma_i = s_j - s_0 + \pi(K_B^{-1}) + \sum a_n D_n \,,$$

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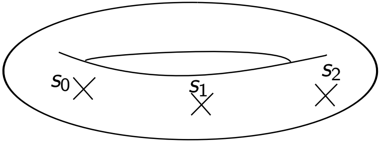

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2. **torsion $\mathbb{Z}_n$** if sections $n \cdot s_i \sim s_0$. Generator of $\mathbb{Z}_n$ quotient symmetries $G = G'/\mathbb{Z}_n$.
   Torsion Shioda map $\sigma_T^{(n)}$ that is a trivial $\mathbb{Q}$-divisor $n \cdot \sigma_T^{(n)} \sim 0$
   Matter curves must have charge $\sigma_T^{(n)} \cdot_m = 0 \text{ mod } n$ [Mayrhover,Morrison,Till,Weigand]

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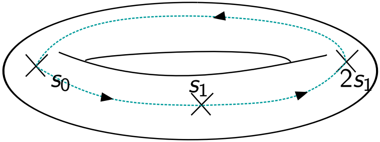

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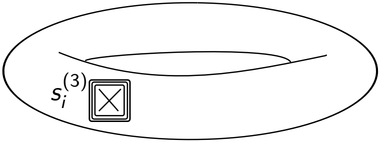
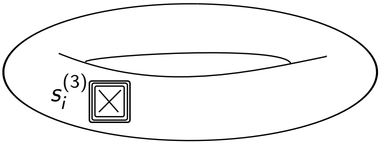


Multi-Sections

- A section with $s_i^{(n)} \cdot \mathcal{E} = n > 1$ is an n-section. If the torus admits only multi-sections it is a genus-one curve.

The Zoo of Sections



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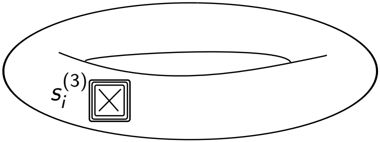

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- The n-section generates a discrete $\mathbb{Z}_n$ symmetry in the effective field theory

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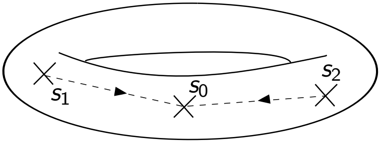

Multi-Sections

- The n-section generates a discrete $\mathbb{Z}_n$ symmetry in the effective field theory
- The n-section can be obtained by $n+1$ collapsing rational sections via a conifold transitions:
  In the effective field theory this is a higgsing: $U(1)^n \rightarrow \mathbb{Z}_n$ [V. Braun, Morrison; Anderson, Garca-Etxebarria,Grimm,Keitel;]

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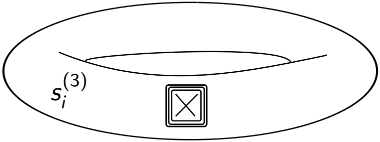

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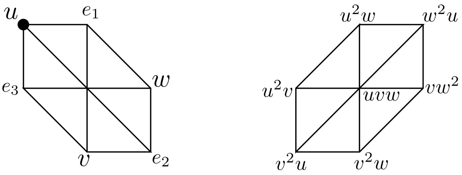
- Why not write down a *discrete Shioda map:* [Klevers, Piragua,Pena, O., Reuter]

$$\sigma_D^{(n)} = s_i^{(n)} + \text{Base} + \sum_i a_i D_i \,.$$

- Matter charges $c_m \cdot \sigma_D^{(n)} = k \mod n$

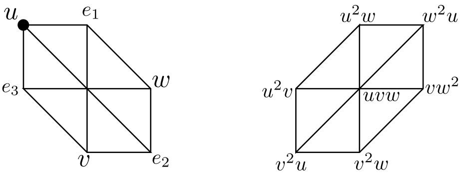
Example: $U(1)^3 \to \mathbb{Z}_3$



- Consider a polytope Δ and its polar-dual Δ^o

$$P_\Delta = \sum_{m \in \Delta^o} \prod_{i=1}^{n} s_m z_i^{\langle m, \rho_i \rangle + 1} = 0 \,,$$

Example: $U(1)^3 \to \mathbb{Z}_3$



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$$P_\Delta = \sum_{m \in \Delta^o} \prod_{i=1}^{n} s_m z_i^{\langle m,\, \rho_i \rangle + 1} = 0\,,$$

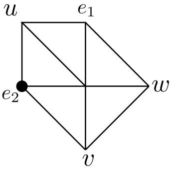
- Start from other ambient spaces such as dP_3:

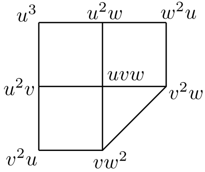
$$\begin{aligned} p_{dP_3} =& s_2 e_1 e_3^2 u^2 v + s_3 e_2 e_3^2 u v^2 + s_5 e_1^2 e_3 u^2 w + s_6 e_1 e_2 e_3 uvw + s_7 e_2^2 e_3 v^2 w \\ &+ s_8 e_1^2 e_2 u w^2 + s_9 e_1 e_2^2 w^2 v\,. \end{aligned}$$

with zero-section $u \cup p_{dP_3} = 0$ and three sections $e_i \cup p_{dP_3} = 0$

$\boxed{U(1)^3}$ theory

Example: $U(1)^3 \to \mathbb{Z}_3$





- Consider a polytope Δ and its polar-dual Δ^o

$$P_\Delta = \sum_{m\in\Delta^o} \prod_{i=1}^{n} s_m z_i^{\langle m,\, \rho_i\rangle + 1} = 0\,,$$

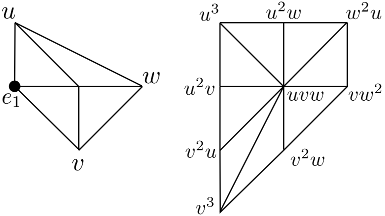
- Collapsing the sections to dP_2

$$\begin{aligned} p_{dP_2} = & s_1 e_2^2 e_1^2 u^3 + s_2 e_2^2 e_1 u^2 v + s_3 e_2^2 u v^2 + s_5 e_2 e_1^2 u^2 w + s_6 e_2 e_1 uvw \\ & + s_7 e_2 v^2 w + s_8 e_1^2 u w^2 + s_9 e_1 v w^2\,, \end{aligned}$$

with zero-section $u \cup p_{dP_2} = 0$ and two sections $e_i \cup p_{dP_2} = 0$

$\boxed{U(1)^2}$ theory

Example: $U(1)^3 \to \mathbb{Z}_3$



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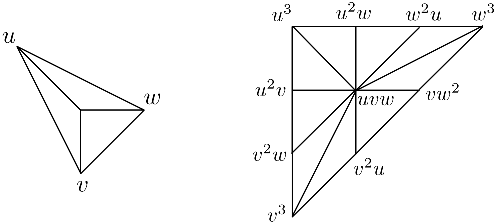
- Collapsing the sections to dP_1

$$p_{P^2} = s_1 u^3 e_1^2 + s_2 u^2 v e_1^2 + s_3 u v^2 e_1^2 + s_4 v^3 e_1^2 + s_5 u^2 w e_1 + s_6 uvw e_1 + s_7 v^2 w e_1 + s_8 u w^2 + s_9 v w^2\,,$$

with zero-section $e_1 \cup p_{dP_3} = 0$ and *non-toric* section

$\boxed{U(1)^1}$ theory

Example: $U(1)^3 \to \mathbb{Z}_3$



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- Collapsing the sections to P^2

$$\begin{aligned} p_{dP_1} =& s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3 + s_5 u^2 w + s_6 uvw \\ &+ s_7 v^2 w + s_8 u w^2 + s_9 v w^2 + s_{10} w^3\,, \end{aligned}$$

with three-sections only $(u, v, w) \cup p_{P^2} = 3$

$\boxed{\mathbb{Z}_3}$ theory

Take Home Message

- Multi-Sections in genus one curves can be though of as **collapsed rational sections** generating a free Mordell-Weil group
- Rational Sections can be obtained by non ADE blow-ups of the fiber ambient space
- If you look for sections, go search in other fiber ambient spaces

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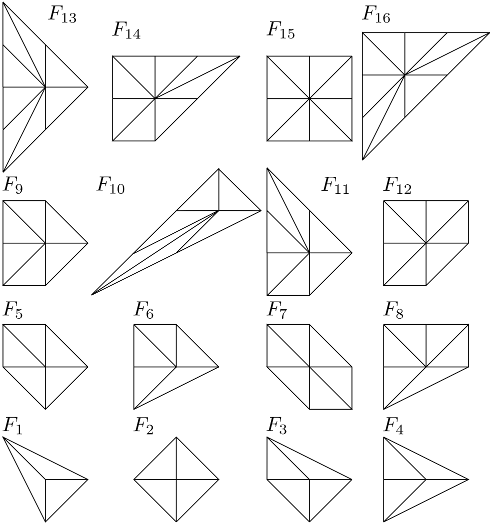
- Rational Sections can be obtained by non ADE blow-ups of the fiber ambient space
- If you look for sections, go search in other fiber ambient spaces

[Kreutzer, Skarke]

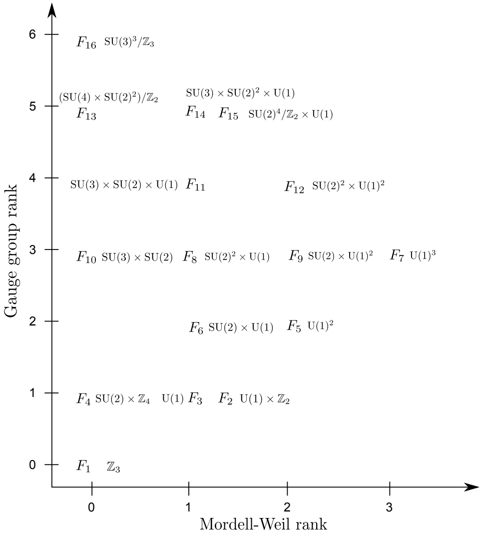
- In two dimension there are only 16 of them!
- In three dimensions there are only 3145 of them!

Outline

The two dimensional Case



The two dimensional Network



Torsion

- $\mathbb{Z}_2, \mathbb{Z}_3$ torsion in the upper theories

Multi Sections

- Genus one curves with two-and three-sections

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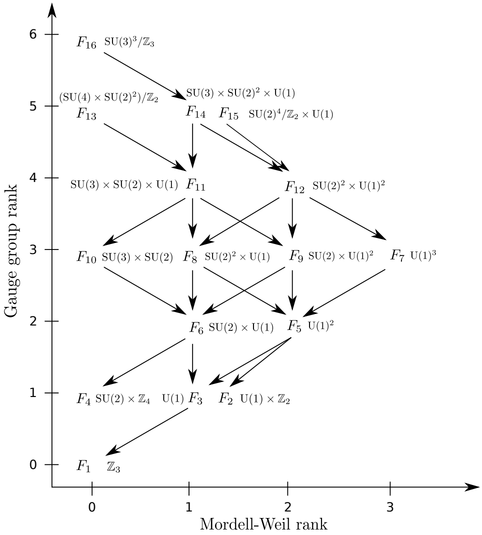

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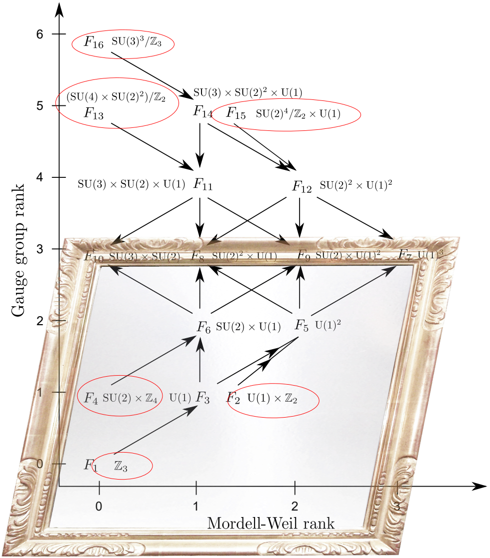

Torsion

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Mirror Duality

- $MW(P_\Delta) = MW(P_{\Delta^\circ})$
- TorMW $\leftrightarrow$ n-Sections

Multi Sections

- Genus one curves with two-and three-sections

Understanding the Duality

- Can we relate the appearance of torsion directly from properties of the ambien polytope?
- Can we relate the appearance of (only) multi-sections directly from properties of the ambien polytope?
- Are they connected by $\Delta \leftrightarrow \Delta^o$?
- Does this generalize?

Torsion and Multi-Sections from lattice refinement

- Remember: Lattice points $\rho_i \in \Delta$ correspond to divisors D_i. From a dual lattice point m we can obtain linear equivalence relations betwen divisors D_i

$$\sum_i \langle \rho_i, m \rangle D_i \sim 0 \,.$$

Torsion and Multi-Sections from lattice refinement

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$$\sum_i \langle \rho_i, m \rangle D_i \sim 0\,.$$

- The torsion Shioda map is a k-trivial divisor: [Mayrhover,Till,Morrison,Weigand]

$$k \cdot \sigma_T^{(k)} \sim 0$$

Torsion and Multi-Sections from lattice refinement

- Remember: Lattice points $\rho_i \in \Delta$ correspond to divisors D_i. From a dual lattice point m we can obtain linear equivalence relations betwen divisors D_i

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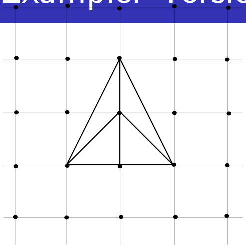
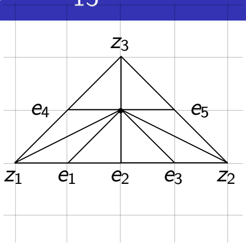
- The vertices ρ_i^{ver} span a lattice of finite index k in Δ
- As Δ is of index k, the dual lattice M can be refined by a factor of k.
- The torsion Shioda map is the divisor obtained from the refined dual lattice points

Example: Torsion from F_{13}





Obtaining the Torsion Shioda map

- The F_{13} fiber has a generic $SU(4) \times SU(2)^2/\mathbb{Z}_2$ gauge symmetry that are resolved by the e_i divisors

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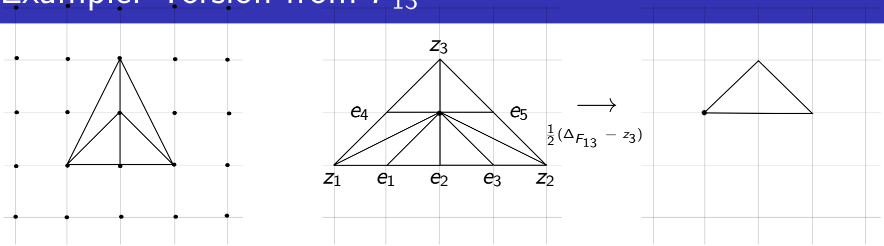

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- The F_{13} fiber has a generic $SU(4) \times SU(2)^2/\mathbb{Z}_2$ gauge symmetry that are resolved by the e_i divisors
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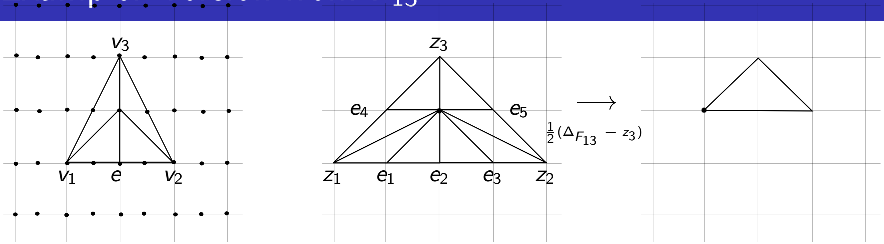

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- There exists a refined dual lattice $M \to M'$ with i.e. $m' = (\frac{1}{2}, 0)$
- Construct the torsion Shioda map from refined lattice point

$$\sum_i \langle m', \rho_i \rangle D_i = \sigma_t^{(2)} = [z_2] - [z_1] + \frac{1}{2}(-[e_1] + [e_3] + [e_4] - [e_5]) \checkmark$$

[Mayrhover, Till, Morrison, Weigand]

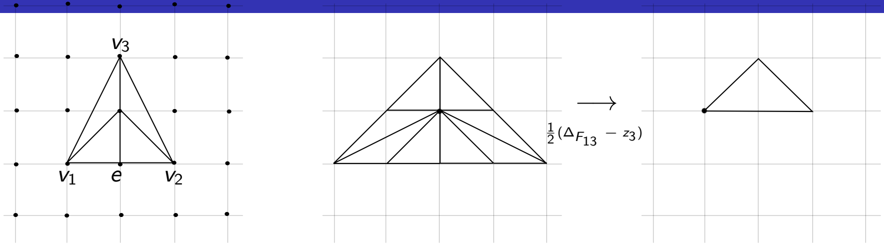
Torsion and Multi-Sections from lattice refinement

Discrete Shioda map from lattice refinement

Now we reverse the argument: We have assumed the vertices of $\rho_i^{(ver)}$ of Δ span a lattice of index k in N

- Now $\Delta \to \Delta^o$ is a lattice polytope of index k in M that corresponds to a divisor i.e. the anticanonical divisor
- The divisor class constructed from polytope Δ^o is shift invariant $[D_{\Delta^o+v}] = [D_{\Delta^o}]$
- Scaling a polytope Δ^o scales the divisor class $[D_{k\Delta^o}] = [kD_{\Delta^o}]$
- However the divisor D_{Δ^o} corresponds a divisor in the anticannonical class of the ambient space.
- Hence the anti-cannonical class is a k-multiple of an integral class because by assumption is spans a lattice of index k
- Intersections with the class of the vertices in Δ are therefore a k-multiple only

Example: Multi-Sections from F_4



Obtaining the Torsion Shioda map

We have already seen that we can shrink the dual polytope to $\Delta_{F_{13}} \rightarrow \Delta' = \frac{1}{2}(\Delta - w)$

- Within F_4 ,F_{13} describes exactly the anticanonical divisor in

$$-K = [v_1] + [v_2] + [v_3] + [e] \sim 2 \underbrace{(2[v_1] + e)}_{D_{\Delta'}}$$

- The fiber in F_4 is a genus one curve only with 2-sections i.e. a theory with $SU(2) \times \mathbb{Z}_2$ gauge symmetry

The mirror Conjecture

- This procedure holds true for all 16 fibers constructed from 2D ambient spaces:

Conjecture 1. *Given a genus-one fiber $\mathcal{C}$ for which the Mordell-Weil group of the Jacobian contains torsion, the mirror dual is a genus-one fiber $\mathcal{C}'$ without a section and vice versa.*

The mirror Conjecture

Is this phenomenon restricted to fibers in 2D ambient spaces?

Specialities in 2D

- Vertices have positive intersection with the elliptic curve
- Divisors are also curves
- Singularities are torically resolved
- $\mathrm{Vol}(\Delta) + \mathrm{Vol}(\Delta^o) = const$
- $\#Points(\Delta) + \#Points(\Delta^o) = 12$

- Is there any reason why this should hold in general?
- In complete intersection fibers, the above constraints do not hold!

Outline

Recap: Fibers in 3D ambient spaces

- A complete intersection Calabi-Yau is described in terms of a *nef* partition of a polytope a d-dimensional polytope Δ

$$\begin{aligned} \Delta &= \Delta_1 + ... + \Delta_n\,, \qquad & \Delta^\circ &= \langle \nabla_1, ..., \nabla_n \rangle\,, \\ \nabla^\circ &= \langle \Delta_1, ..., \Delta_n \rangle\,, \qquad & \nabla &= \nabla_1 + ... + \nabla_n, \end{aligned} \tag{1}$$

- This specifies a codimension n Calabi-Yau in a d-dimensional polytope via the intersections of

$$P_{\Delta_i} = \sum_{m \in \Delta_i} \prod_{i=1}^{n} s_m z_i^{\langle m,\, \rho_i \rangle + a_i} \in \mathbb{P}_{\Delta^o}$$

  with $a_i \geq -1$
- The Mirror CY is cut out by $P_{\nabla_i} \in \mathbb{P}_{\nabla^o}$
- Note: One ambient space can have multiple nef partitions whose mirror dual do not have to live in the same ambient spaces!

How to check it?

The arguments in 2D do not readily apply. The existence of torsion does not imply multisections in the dual geometry anymore!
In [Braun, Grimm, Keitel '15] 3145 codimension two nef partitions in 3D ambient spaces have been considered

- They constructed *the toric MW group*
- They mapped all curves into their Jacobian Form i.e. into WSF form

A new hope:

- The fiber in $\mathbb{P}^3$ only admits four sections
- The dual Nef parition with Palp id $(3145, 0)$ has $\mathbb{Z}_4$ Mordell-Weil torsion

Conjecture counter: 2/3145

Does it generalize?

- We went through the full list again and obtained all intersections of *toric divisors* with the elliptic curve
- Combining this information with the Mordell-Weil group, we indeed get a match in

Conjecture counter: 3086/3145

- So what is wrong about the rest?

Reducibility and non-toric sections

Nef partition (4, 0)

Specified by the nef partition

$$\nabla_1 = \langle z_0,\, z_3,\, z_4,\, z_6 \rangle\,, \quad \nabla_2 = \langle z_1,\, z_2,\, z_5,\, z_8 \rangle\,,$$

in the $\mathbb{P}^1 \times \mathbb{P}^2$ ambient space:

z_0	z_1	z_2	z_3	z_4
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

- Two inequivalent divisor classes: $[z_0] \sim [z_1]\,, \quad [z_2] \sim [z_3] \sim [z_4]$
- Intersections $\mathcal{E} \cdot [z_0] = 2$ and $\mathcal{E} \cdot [z_2] = 3$.
- The dual nef partition $(3013, 1)$ has no torsion.
- This is not a genus one curve: Construct a non-toric zero-section [Braun, Grimm, Keitel 14]

$$[s_0] = [z_2] - [z_0] + H_B$$

Reducibility and multi-mirrors

- Lets consider the elliptic curve with CICY equation of (3013, 1) in z_i

$$
\begin{aligned}
p_1 = & a_0 z_5 z_6 z_7 z_8 z_9 z_{10} z_{11} z_{12} z_{13} z_{15} z_{16} + a_2 z_0 z_3 z_5^2 z_7^2 z_{11} z_{12} z_{13} \\
& + a_1 z_1 z_4 z_6^2 z_8^2 z_{13} z_{15} z_{16} + a_3 z_0 z_1 z_2 z_5 z_6 z_9 z_{12} z_{16} \\
p_2 = & a_5 z_2 z_9^2 z_{10}^2 z_{11} z_{12} z_{15} z_{16} + a_4 z_3 z_4 z_7 z_8 z_{10} z_{11} z_{13} z_{15} + a_6 z_0 z_1 z_2 z_3 z_4
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&+ a_1 z_1 z_4 z_6^2 z_8^2 z_{13} z_{15} z_{16} + a_3 z_0 z_1 z_2 z_5 z_6 z_9 z_{12} z_{16} \\
p_2 =& a_5 z_2 z_9^2 z_{10}^2 z_{11} z_{12} z_{15} z_{16} + a_4 z_3 z_4 z_7 z_8 z_{10} z_{11} z_{13} z_{15} + a_6 z_0 z_1 z_2 z_3 z_4
\end{aligned}
$$

- Lets also consider the elliptic curve with CICY equation of (3013,0) in z_i

$$
\begin{aligned}
p_1 =& a_0 z_5 z_6 z_7 z_8 z_9 z_{10} z_{11} z_{12} z_{13} z_{15} z_{16} + a_2 z_0 z_3 z_5^2 z_7^2 z_{11} z_{12} z_{13} \\
&+ a_1 z_1 z_4 z_6^2 z_8^2 z_{13} z_{15} z_{16} + a_3 z_0 z_1 z_2 z_5 z_6 z_9 z_{12} z_{16} \\
p_2 =& a_5 z_2 z_9^2 z_{10}^2 z_{11} z_{12} z_{15} z_{16} + a_4 z_3 z_4 z_7 z_8 z_{10} z_{11} z_{13} z_{15} + a_6 z_0 z_1 z_2 z_3 z_4
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- As they come from the same ambient space, they also have exactly the same Stanley-Reisner ideal: These are equivalent elliptic curves!

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- Both curves have a zero-section and no torsion

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- As they come from the same ambient space, they also have exactly the same Stanley-Reisner ideal: These are equivalent elliptic curves!
- Both curves have a zero-section and no torsion
- However, the first has (4, 0) as a mirror dual, the second one (5, 1). Are those equivalent too?

Reducibility

- Nef (5, 1) admits a toric section with CICY equation in z_i

$$
\begin{aligned}
p_1 =& a_8 z_0 z_2^2 + a_7 z_0 z_2 z_3 + a_5 z_0 z_3^2 + a_6 z_0 z_2 z_4 + a_4 z_0 z_3 z_4 \\
&+ a_3 z_0 z_4^2 + a_2 z_1 z_2 + a_1 z_1 z_3 + a_0 z_1 z_4 \\
p_2 =& a_{17} z_0 z_2^2 + a_{16} z_0 z_2 z_3 + a_{14} z_0 z_3^2 + a_{15} z_0 z_2 z_4 \\
&+ a_{13} z_0 z_3 z_4 + a_{12} z_0 z_4^2 + a_{11} z_1 z_2 + a_{10} z_1 z_3 + a_9 z_1 z_4
\end{aligned}
$$

- The CICY equation of (4,0) is very different and has no toric section:

$$
\begin{aligned}
p_1 =& a_{11} z_0 z_2^2 + a_5 z_1 z_2^2 + a_{10} z_0 z_2 z_3 + a_4 z_1 z_2 z_3 + a_8 z_0 z_3^2 + a_2 z_1 z_3^2 \\
&+ a_9 z_0 z_2 z_4 + a_3 z_1 z_2 z_4 + a_7 z_0 z_3 z_4 + a_1 z_1 z_3 z_4 + a_6 z_0 z_4^2 + a_0 z_1 z_4^2 \\
p_2 =& a_{17} z_0 z_2 + a_{14} z_1 z_2 + a_{16} z_0 z_3 + a_{13} z_1 z_3 + a_{15} z_0 z_4 + a_{12} z_1 z_4
\end{aligned}
$$

- Those models share the same Weierstrass models with:

$$
(5,1) : f(a_i), g(a_i) \quad = \quad (4,0) : f(a_i), g(a_i)
$$

  After relabeling the sections a_i!
- Hence the curve in (4, 0) is **equivalent** to (5,1)

Conjecture counter: 3088/3145

Reducibility of CICY fibers

- By this procedure we find that many fibers have an **equivalent (singular) Weierstrass** description related by a simple **relabeling of the sections**
- The 3145 different nef partitions get reduced to 1024 inequivalent fibers
- We find examples where ADE singularities have toric and non-toric resolutions when realized in different ambient spaces

Conjecture counter: 998/1024

Mirror genus-one curves

26 cases to go

Conjecture counter: 998/1024

- In all cases, the curve and its dual are genus-one curves that only admit multi-sections. $\rightarrow$ No toric Mordell-Weil group. How to test for the torsion?
- We consider the Weiertrass models of those curves (Jacobian) and and show that they are birational equivalent to the Weierstrass from with k-torsion

  [Aspinwall, Morrison '98]

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  [Aspinwall, Morrison '98]
- Those models have torsion sections only in their Jacobian
- In 24 cases the genus one curve has only two-sections and two-torsion, just as their mirrors ✓

$\leftarrow$ mirror dual	$\rightarrow$	$\leftarrow$ mirror dual	$\rightarrow$
(152, 0)	(195, 4)	(8, 0)	(609, 0)
(29, 2)	(577, 0)	(34, 0)	(321, 1)
(39, 0)	(335, 0)	(56, 2)	(356, 2)
(78, 2)	(266, 0)	(108, 0)	(161, 1)
(129, 0)	(129, 1)	(150, 1)	(208, 1)
(152, 1) self-mirror		(122, 0) self-mirror	

Mirror genus-one curves

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- We consider the Weiertrass models of those curves (Jacobian) and and show that they are birational equivalent to the Weierstrass from with k-torsion [Aspinwall, Morrison '98]
- in four cases, the degree of the Multi-sections and torsion does not match!

$\leftarrow$ mirror dual $\rightarrow$

nef (5, 3)	nef (2069, 0)
four-sections	$\mathbb{Z}_2$ torsion
no torsion	one-sections

nef (21, 1)	nef (488, 0)
four-sections	$\mathbb{Z}_2$ torsion
$\mathbb{Z}_2$ torsion	two-sections

- These theories have non-toric resolution divisors and are connected via a higgsing.
- Apart from the matching of the degree, the conjecture still holds

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[Aspinwall, Morrison '98]

Conjecture counter: 1024(4)/1024 ✓

Mirror genus-one curves

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Conjecture 2. *A curve $\mathcal{C}$ constructed as a complete intersection in a toric ambient space such that all the codimension one loci are torically resolved does exhibit Mordell-Weil torsion of degree k in the Jacobian if and only if the one dimensional generators of the fan $\{\rho_i \,:\, D_i \cdot C \neq 0\}$ span a sublattice $\tilde{M} \supset M$ of index k. Up to base divisors a point $m \in \tilde{M} \backslash M$ corresponds to a torsion Shioda map $\sigma_t^{(k)}$ via*

$$\sigma_t^{(k)} = \sum_{\rho_i \in \Sigma(1)} \langle m,\, \rho_i \rangle \cdot D_i \,.$$

Outline

A self-dual genus one-curve

- The nef partition is given by:

$$\nabla_1 = \langle z_0,\, z_3,\, z_4,\, z_6 \rangle\,, \quad \nabla_2 = \langle z_1,\, z_2,\, z_5,\, z_8 \rangle\,,$$

- in the ambient space

z_0	z_1	z_2	z_3
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

z_4	z_5	z_6	z_8
$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

- The CICY equations are given as

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6\,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8\,.$$

- Intersecting the curve: $[z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_8] \cdot \mathcal{E} = [0, 0, 2, 2, 2, 2, 0, 0]$
- We have only-two sections: this is a genus-one curve
- The vertices span a lattice of index 2 in $\mathbb{Z}^3$

Gauge symmetry

z_0	z_1	z_2	z_3
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

z_4	z_5	z_6	z_8
$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

- The dual lattice is generated by a finer lattice i.e. $m' = (\frac{1}{2}, \frac{1}{2}, 0), m_1 = (0,1,0), m_2 = (0,0,1)$
- From m' we construct the torsion Shioda map:

$$\sigma_T^{(2)} = [z_5] - [z_3] + \frac{1}{2}\left([z_0] + [z_1] - [z_6] + [z_8]\right) .$$

- Indeed: z_0, z_1, z_6, z_8 are resolution divisors
- The intersection with matter $\sigma_T^{(2)} \cdot c_m \sim 0$ implies for the weights λ^i

$$\lambda_m^1 + \lambda_m^2 - \lambda_m^3 + \lambda_m^4 = 0 \text{ mod } 2$$

- Only bifundemantal matter possible

Consistency of torsion

We can check for the existence of $\mathbb{Z}_2$ torsion in the Weierstrass form:

- A Weierstrass model with a $\mathbb{Z}_2$ torsion points admits the following form of Weierstrass coefficients: [Aspinwall, Morrison' 98]

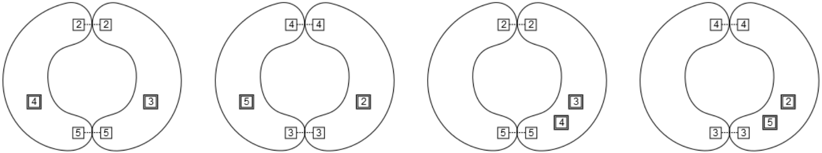
$$f = A_4 - \frac{1}{3}A_2^2\,, \quad g = \frac{1}{27}A_2(2A_2^2 - 9A_4)\,,$$
$$\Delta = A_4^2(4A_4 - A_2^2)\,,$$

- The $(122, 0)$ Weierstrass coefficients are related by the birational map:

$$\begin{aligned} A_2 \;\to\;& 4a_1a_4a_5a_6 + a_3^2a_6^2 - 2a_1a_3a_6a_7 + a_1^2a_7^2 \\ & - 4a_0a_2a_7^2 - 4a_1^2a_5a_8 + 16a_0a_2a_5a_8 \\ & - 8a_0a_4a_5a_9 + 4a_0a_3a_7a_9\,, \\ A_4 \;\to\;& 16a_0a_5(a_4^2a_5 - a_3a_4a_7 + a_3^2a_8) \\ & \cdot (a_2a_6^2 - a_1a_6a_9 + a_0a_9^2)\,. \end{aligned}$$

- Therefore we have $\mathbb{Z}_2$ torsion ✓
- We have four SU(2) singularities in codimension two ✓

Discrete Shioda map and gauge locus

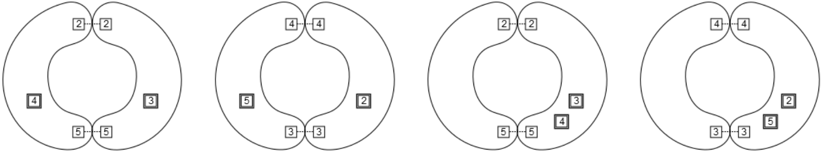


- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \,.$$

- We have the following four codimension one, I_2 singularities:

Discrete Shioda map and gauge locus



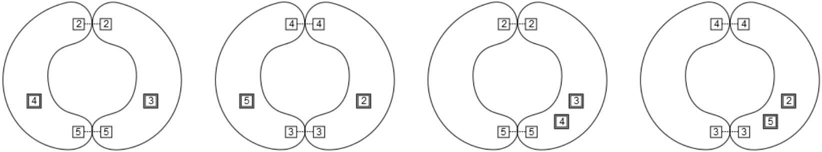
- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \,.$$

- We have the following four codimension one, I_2 singularities:
- toric locus I_2 locus L_1 : $a_0 = 0$

$$p_1^{(1)} = p' \cdot z_0 \,,$$

Discrete Shioda map and gauge locus



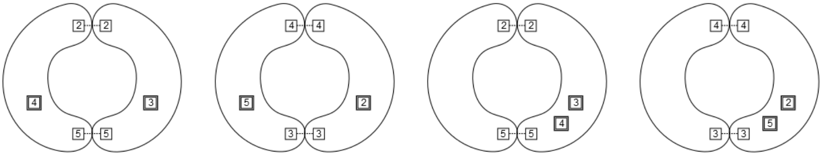
- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \,.$$

- We have the following four codimension one, I_2 singularities:
- toric locus I_2 locus L_2 : $a_5 = 0$

$$p_2 = p_2' \cdot z_1 \,.$$

Discrete Shioda map and gauge locus



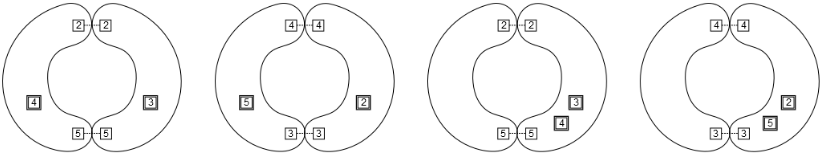
- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6\,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8\,.$$

- We have the following four codimension one, I_2 singularities:
- non-toric I_2 locus L_3 : $a_4^2 a_5 - a_3 a_4 a_7 + a_3^2 a_8 = 0$ the following combination factorizes:

$$a_3 a_4 z_0 z_5 p_2 - (a_4 a_5 z_2 + a_3 a_8 z_1 z_5) p_1 = p^{(2)} \cdot z_6$$

Discrete Shioda map and gauge locus



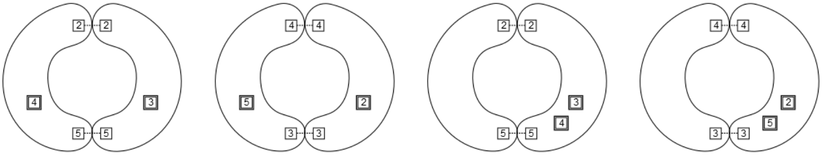
- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \,.$$

- We have the following four codimension one, I_2 singularities:
- non-toric I_2 locus L_4 : $a_2 a_6^2 - a_1 a_6 a_9 + a_0 a_9^2 = 0$ the following combination factorizes:

$$a_6 a_9 z_1 z_4 p_1 - (a_0 a_9 z_3 + a_2 a_6 z_0 z_4) p_2 = p^{(3)} \cdot z_8 \,,$$

Discrete Shioda map and gauge locus



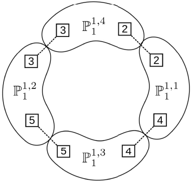
- The CICY equations are

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \, ,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \, .$$

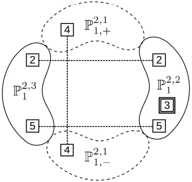
- We have the following four codimension one, I_2 singularities:
- The multisections intersect the fiber and hence we have to orthogonolize the discrete Shioda map;

$$\sigma^{(2)}_{D,4} = [z_4] + \frac{1}{2}\left([z_1] + [z_6]\right)$$

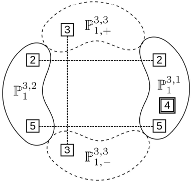
The matter spectrum



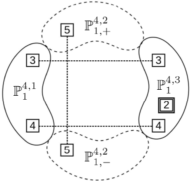
Matter locus 1,
$a_0 = a_5 = 0$



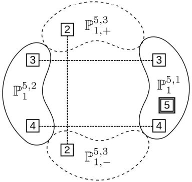
Matter locus 2,
$a_0 = a_6 = 0$



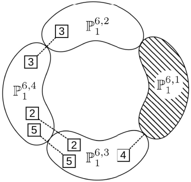
Matter locus 3,
$a_0 = a_2a_6 - a_1a_9 = 0$



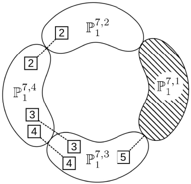
Matter locus 4,
$a_5 = a_3 = 0$



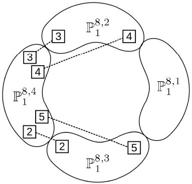
Matter locus 5,
$a_5 = a_4a_7 - a_3a_8 = 0$



Matter locus 6,
$a_0 = 0, L_2$



Matter locus 7,
$a_5 = 0, L_3$



Matter locus 8,
L_2, L_3

The matter spectrum

Locus	(f, g, Δ)	$SU(2)^4 \times \mathbb{Z}_2$ Rep.
$a_0 = 0\,,\ a_5 = 0$	$(0, 0, 4)$	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})_{\frac{1}{2}}$
$a_0 = 0\,,\ a_6 = 0$	$(0, 0, 4)$	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})_{1}$
$a_0 = 0\,,$ $a_2 a_6 - a_1 a_9 = 0$	$(0, 0, 4)$	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})'_{0}$
$a_5 = 0\,,\ a_3 = 0$	$(0, 0, 4)$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_{0}$
$a_5 = 0\,,$ $a_4 a_7 - a_3 a_8 = 0$	$(0, 0, 4)$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})'_{1}$
$a_0 = 0\,,$ $a_4^2 a_5 - a_3 a_4 a_7 + a_3^2 a_8 = 0$	$(0, 0, 4)$	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2}}$
$a_5 = 0\,,$ $a_2 a_6^2 - a_1 a_6 a_9 + a_0 a_9^2 = 0$	$(0, 0, 4)$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2}}$
$a_4^2 a_5 - a_3 a_4 a_7 + a_3^2 a_8 = 0\,,$ $a_2 a_6^2 - a_1 a_6 a_9 + a_0 a_9^2 = 0$	$(0, 0, 4)$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})_{\frac{1}{2}}$

The matter spectrum

- Only bifundamental matter ✓
- All matter curves distinguished by a unique quantum number ✓

$$
\begin{aligned}
&(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})_1\,, \quad (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})'_0 \\
&(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_1\,, \quad (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})'_0
\end{aligned}
$$

- The $\mathbb{Z}_2$ also restricts the Yukawa couplings as expected

$$
\begin{aligned}
Y_1 &: (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})_{\frac{1}{2}} \cdot (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2}} \cdot (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_0\,, \\
Y_2 &: (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})_{\frac{1}{2}} \cdot \overline{(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2}}} \cdot (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})'_1\,.
\end{aligned}
$$

- Without the $\mathbb{Z}_2$ symmetry the geometric different Yukawa couplings would be the same

Summary and Outlook

- We have given strong evidence, that genus-one curves with multi-sections are mirror-dual to fibers with Mordell-Weil torsion of the same degree
- Combinatorial explanation in 2D toric ambient spaces
- We have explicitly checked the conjecture for **all 3145 cases** of codimension two curves
- The combinatorial explanation does not fully carry over to 3D
- We find fibers with new features:
    - Equivalent realizations of the same elliptic curve
    - A fiber with a non-toric zero-section
    - Genus-one fibers with torsion sections in their Jacobian
- We have fully analyzed a self-dual genus one curve that admits quotient and discrete symmetries

Outlook

- Can we proof the conjecture in general?
- Is there a physical explanation?
- Can we classify discrete symmetries in F-theory via their mirror dual torsion?