Noninvertible Gauss Law and Axions

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Based on [**YC**-Lam-Shao 2212.04499]
(See also [Yokokura 2212.05001])

Symmetry and Topological Operators

- Global symmetries and their 't Hooft anomalies are important nonperturbative tools to study dynamics of QFTs, e.g. constraints on RG flows.
- The notion of a global symmetry has been significantly generalized in recent years.
- Simply put, any **topological operator** in a given QFT is viewed as a generalized symmetry operator.

Symmetry and Topological Operators

- A global symmetry manifests itself as the existence of conserved quantities.
- For example, consider a 4d QFT with a $U(1)$ symmetry with the conserved Noether current $j_\mu(x)$, $\partial^\mu j_\mu = 0 \Leftrightarrow d \star j = 0$.
- Due to the conservation equation, the symmetry operator

$$U_\alpha = \exp\left(i\alpha \int_{\text{space}} d^3x\, j^0\right) , \quad \alpha \in [0, 2\pi)$$

is conserved under time evolution.

Symmetry and Topological Operators

$$U_\alpha = \exp\left(i\alpha \int_{\text{space}} d^3x\, j^0\right) .$$

- It is more natural to define the symmetry operator/defect, $U_\alpha(M)$, which is supported on an arbitrary closed, codimension-1 submanifold M inside the spacetime

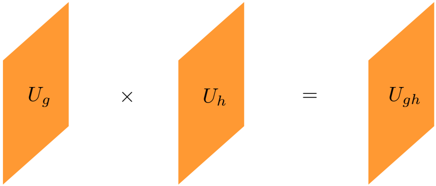
$$U_\alpha(M) = \exp\left(i\alpha \oint_M \star j\right)$$

  instead of picking a particular spatial slice.

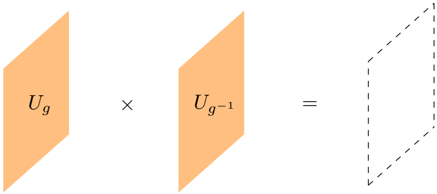
- Thanks to the conservation equation $d \star j = 0$, the symmetry operator $U_\alpha(M)$ becomes a topological operator/defect, i.e. M can be deformed in arbitrary directions in spacetime without changing the value of a correlation function.

Fusion of Symmetry Operators

- We can define the parallel **fusion** between these symmetry operators/defects.



- The fusion algebra of symmetry operators follows the group multiplication law. In particular, every symmetry operator has an **inverse**.

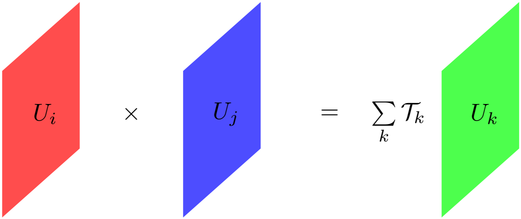


Generalized Symmetries

- We learn: **Symmetry** = Invertible codimension-1 topological operators/defects.
- Being topological is the key property making the symmetries useful, e.g. scale invariant.
- We can try to relax the other two conditions:
  - Topological operators with no inverse: Noninvertible symmetries [Petkova-Zuber 2000; Fuchs-Runkel-Schweigert 2002;...; Bhardwaj-Tachikawa 2017; Chang-Lin-Shao-Wang-Yin 2018; Thorngren-Wang 2019, 2021;...]
  - Topological operators of codimension > 1: Higher-form symmetries [Hellerman-Henriques-Pantev-Sharpe 2006;...;Gaiotto-Kapustin-Seiberg-Willett 2014;...]

Noninvertible Symmetries

- Not every topological operator in QFTs is invertible.
- Prototypical examples: Verlinde lines in 2d rational CFTs.
  [Petkova-Zuber 2000]
- A generalized symmetry generated by **noninvertible** topological operators is called a noninvertible symmetry.
  [Petkova-Zuber 2000; Fuchs-Runkel-Schweigert 2002;...; Bhardwaj-Tachikawa 2017; Chang-Lin-Shao-Wang-Yin 2018; Thorngren-Wang 2019, 2021;...]



In last few years, a lot of examples of noninvertible symmetries in $d = 4$ and higher have been discovered.

[Koide-Nagoya-Yamaguchi 2021; Kaidi-Ohmori-Zheng 2021; **YC**-Córdova-Hsin-Lam-Shao 2021; Roumpedakis-Seifnashri-Shao 2022; Bhardwaj-SchaferNameki-Tiwari 2022; Córdova-Ohmori-Rudelius 2022; AriasTamargo-Rodriguez-Gomez 2022; Hayashi-Tanizaki 2022; **YC**-Córdova-Hsin-Lam-Shao 2022; Kaidi-Zafrir-Zheng 2022; **YC**-Lam-Shao 2022; Córdova-Ohmori 2022; Antinucci-Galati-Rizi 2022; Bashmakov-DelZotto-Hasan 2022; Damia-Argurio-Tizzano 2022; Damia-Argurio-GarciaValdecasas 2022; Moradi-Moosavian-Tiwari 2022; **YC**-Lam-Shao 2022; Bhardwaj-SchaferNameki-Wu 2022; Bartsch-Bullimore-Ferrari-Pearson 2022; Lin-Robbins-Sharpe 2022; GarcíaEtxebarria 2022; Apruzzi-Bah-Bonetti-SchaferNameki 2022; Delcamp 2022; Heckman-Hübner-Torres-Zhang 2022; many many more]

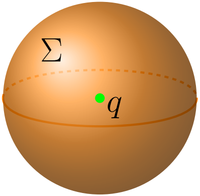
[also Ji-Wen 2019; Ji-Shao-Wen 2019; Kong-Lan-Wen-Zhang-Zheng 2020; Rudelius-Shao 2020; Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021; Nguyen-Tanizaki-Ünsal 2021; Wang-You 2021; Benini-Copetti-Pietro 2022]

Higher-form Symmetries

- (Ordinary) **Symmetry** = Invertible codimension-1 topological operators/defects.
- A generalized symmetry generated by codimension-$(p+1)$ topological operators is called a *p*-form symmetry.
- They naturally acts on extended operators of dimension p by linking [Gaiotto-Kapustin-Seiberg-Willett 2014], and potentially **also** acts on dimension $> p$ operators as well [Bhardwaj-SchaferNameki 2023].
- On the other hand, a *p*-form symmetry can never act on operators of dimension $< p$.
- In the continuous higher-form symmetry case, they are generated by higher-form conserved currents, $J_{\mu\nu}$, $J_{\mu\nu\rho}$, etc.

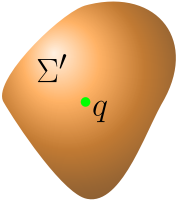
Gauss Law and 1-form Symmetry

- An elementary example of a 1-form symmetry is provided by the Gauss law.
- We surround a probe charge q by a closed surface Σ, and integrate the electric flux through the surface.
(Below, F is normalized such that $\oint F \in 2\pi\mathbb{Z}$.)



$$q = -\frac{i}{e^2} \oint_\Sigma \star F \in \mathbb{Z}$$

Gauss Law and 1-form Symmetry



$$q = -\frac{i}{e^2}\oint_{\Sigma'} \star F \in \mathbb{Z}$$

- Of course, the shape of the surface is not important. The Gauss surface is topological.

- In QFT language, the probe charge is the Wilson line of charge q:

$$W^q \equiv \exp\left(iq \int A \right) .$$

- The Gauss surface defines a topological surface operator:

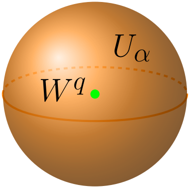
$$Q(\Sigma) \equiv -\frac{i}{e^2}\oint_{\Sigma} \star F .$$

Gauss Law and 1-form Symmetry

- The corresponding symmetry operator is obtained by exponentiating the charge as usual:

$$U_\alpha(\Sigma) \equiv e^{i\alpha Q(\Sigma)}\,, \quad \alpha \in [0, 2\pi)\,.$$

- Topological surface operator in 4d = 1-form symmetry.
- Such a generalized symmetry measuring electric charge of a Wilson line is usually called an electric 1-form symmetry



$= \bullet \times \exp(i\alpha q)$

Axion-Maxwell Theory

Axion-Maxwell Theory

$$\mathcal{L} = \frac{f^2}{2} d\theta \wedge \star d\theta + \frac{1}{2e^2} F \wedge \star F - \frac{i}{8\pi^2} \theta F \wedge F \, .$$

- $\theta \sim \theta + 2\pi$ is the periodic axion field, and $F = dA$ where A is the $U(1)$ electromagnetic gauge field.
- This **axion-Maxwell theory** (i.e. $U(1)$ gauge group) is the simplest axion model that one could write down, and it has many generalized global symmetries. [Hidaka-Nitta-Yokokura 2020 x 2; Brennan-Córdova 2020; Ohmori-Córdova 2022; **YC**-Lam-Shao 2022; Yokokura 2022]
- It provides us with a nice toy example to study generalized global symmetries.

Symmetries of Axion-Maxwell Theory

- There are various current operators of interest:

$$J^{(2)}_{\text{electric}} = -\frac{i}{e^2}F\,, \qquad d\star J^{(2)}_{\text{electric}} = \frac{1}{4\pi^2}d\theta\wedge F\,,$$

$$J^{(2)}_{\text{magnetic}} = \frac{1}{2\pi}\star F\,, \qquad d\star J^{(2)}_{\text{magnetic}} = 0\,,$$

$$J^{(3)}_{\text{winding}} = \frac{1}{2\pi}\star d\theta\,, \qquad d\star J^{(3)}_{\text{winding}} = 0\,.$$

- Let us examine them in view of generalized global symmetries.

  [There is also the shift symmetry for the axion (i.e. Peccei-Quinn) that we will not discuss today.]

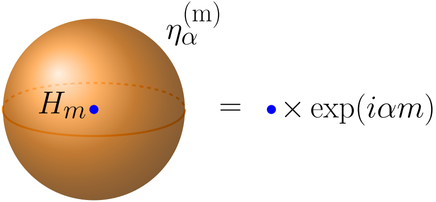
Magnetic 1-form Symmetry

$$J^{(2)}_{\text{magnetic}} = \frac{1}{2\pi} \star F \,, \quad d \star J^{(2)}_{\text{magnetic}} = 0 \,.$$

- The current $J^{(2)}_{\text{magnetic}}$ generates the magnetic 1-form symmetry.
- The corresponding topological symmetry operators are

$$\eta^{(\text{m})}_{\alpha}(\Sigma^{(2)}) \equiv \exp\left(i\alpha \oint_{\Sigma^{(2)}} \frac{F}{2\pi} \right) \,, \quad \alpha \in [0, 2\pi) \,.$$

- Charged objects are 1-dimensional 't Hooft lines H_m.



Winding 2-form Symmetry

$$J^{(3)}_{\text{winding}} = \frac{1}{2\pi} \star d\theta \,, \quad d \star J^{(3)}_{\text{winding}} = 0 \,.$$

- The current $J^{(3)}_{\text{winding}}$ generates the winding 2-form symmetry.
- The corresponding topological symmetry operators are

$$\eta^{(\mathrm{w})}_{\alpha}(\Sigma^{(1)}) \equiv \exp\left(i\alpha \oint_{\Sigma^{(1)}} \frac{d\theta}{2\pi} \right) \,, \quad \alpha \in [0, 2\pi) \,.$$

- Charged objects are 2-dimensional worldsheet of axion strings S_w.

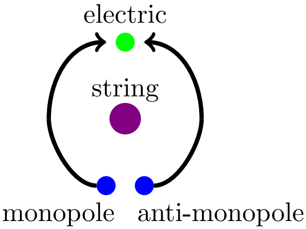
Electric 1-form Symmetry

$$J^{(2)}_{\text{electric}} = -\frac{i}{e^2} F \,, \quad d \star J^{(2)}_{\text{electric}} = \frac{1}{4\pi^2} d\theta \wedge F \neq 0 \,.$$

- The current $J^{(2)}_{\text{electric}}$ would generate the electric 1-form symmetry (a.k.a. Gauss law), but it is **not conserved**!
- The operator $U_\alpha = \exp(\frac{\alpha}{e^2} \oint \star F)$ is thus **not** topological anymore, and it is not possible to measure the electric charge of a probe particle using the ordinary Gauss law.
- Gauss law is "anomalous"!

Anomalous Gauss Law

- Relatedly, there is **no** conserved, quantized, and gauge-invariant charge in the axion-Maxwell theory [Marolf 2000]. Indeed, this is totally expected, due to the Witten effect [Witten 1979].
- Since a monopole carries an electric charge proportional to θ, a monopole going around an **axion string** will freely gain an electric charge, seemingly violating charge conservation. [c.f. Fukuda-Yonekura 2020]



Page Charge

- Let us rewrite the equation of motion:

$$d\left(-\frac{i}{e^2}\star F - \frac{1}{4\pi^2}\theta F\right) = 0\,.$$

- We may define a formally topological operator

$$\hat{U}_\alpha(\Sigma) \equiv e^{i\alpha Q_{\text{page}}}\,, \quad Q_{\text{page}}(\Sigma) \equiv \oint_\Sigma \left(-\frac{i}{e^2}\star F - \frac{1}{4\pi^2}\theta F\right)\,.$$

- In the literature, Q_{Page} is known as Page charge.
  [Page 1983; Marolf 2000]

- Page charge is conserved (topological) and quantized. However, it is **not gauge-invariant**, since it does not respect $\theta \sim \theta + 2\pi$.

Rational Angles

- Can we do any better? For **rational** $\alpha = 2\pi p/N$, we can make progress:

$$\hat{U}_{2\pi p/N} = \exp\left[\oint\left(\frac{2\pi p}{e^2 N} \star F - \frac{ip}{2\pi N}\theta F\right)\right] .$$

- The $-\frac{ip}{2\pi N}\theta F$ term is the source of **non-gauge-invariance**. It is not gauge invariant since the level is not properly quantized ($\oint F \in 2\pi\mathbb{Z}$).

- However, such improperly quantized effective actions are widely used in condensed matter physics. Moreover, there is a well-known way to make it gauge-invariant!

Topological Quantum Field Theory

- Improperly quantized effective actions often arise from the **response of a TQFT to external fields**, such as the low-energy limit of fractional quantum Hall states.
- The $-\frac{ip}{2\pi N}\theta F$ term can be realized as an effective action for the 2d $\mathbb{Z}_N$ gauge theory:

$$-\oint \frac{ip}{2\pi N}\theta F \to \oint \left[\frac{iN}{2\pi}\phi dc + \frac{ip}{2\pi}\theta dc + \frac{i}{2\pi}\phi dA\right] .$$

- $\phi \sim \phi + 2\pi$ and c is a $U(1)$ gauge field. Heuristically, one can integrated out c to obtain "$\phi = -p\theta/N$" and go back to LHS.

Back to Axion-Maxwell

$$\hat{U}_{2\pi p/N} = \exp\left[\oint\left(\frac{2\pi p}{e^2 N}\star F - \frac{ip}{2\pi N}\theta F\right)\right]$$

$$\downarrow$$

$$\mathcal{D}_{p/N} \equiv \int [D\phi][Dc]\exp\left[\oint\left(\frac{2\pi p}{e^2 N}\star F + \frac{iN}{2\pi}\phi dc + \frac{ip}{2\pi}\theta dc + \frac{i}{2\pi}\phi dA\right)\right]$$

- Motivated by this, we define a new surface operator $\mathcal{D}_{p/N}$. ϕ and c are **auxiliary fields** living on $\mathcal{D}_{p/N}$.
  [Analogous to **YC**-Lam-Shao 2022 (2205.05086); Córdova-Ohmori 2022]
- $\mathcal{D}_{p/N}$ is now topological **and** gauge-invariant! It defines a new **1-form symmetry** of the axion-Maxwell theory. [There is an alternative way to rigorously prove that $\mathcal{D}_{p/N}$ is indeed a topological operator via "higher gauging."]

Noninvertible Symmetry

$$\mathcal{D}_{p/N} \equiv \int [D\phi][Dc] \exp\left[\oint \left(\frac{2\pi p}{e^2 N} \star F + \frac{iN}{2\pi}\phi dc + \frac{ip}{2\pi}\theta dc + \frac{i}{2\pi}\phi dA \right) \right]$$

- However, the operator $\mathcal{D}_{p/N}$ does not obey any ordinary group multiplication law. It is a noninvertible symmetry.

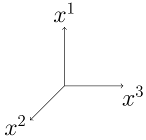
$$\mathcal{D}_{p/N}(\Sigma^{(2)}) \times \overline{\mathcal{D}}_{p/N}(\Sigma^{(2)}) \sim \left(\sum_{n=1}^{N} \eta^{(\mathrm{m})}_{2\pi/N}(\Sigma^{(2)}) \right) \times \left(\sum_{\Sigma^{(1)} \in H_1(\Sigma^{(2)};\mathbb{Z}_N)} \eta^{(\mathrm{w})}_{2\pi p/N}(\Sigma^{(1)}) \right) \neq 1\,.$$

Symmetries of Axion-Maxwell Theory

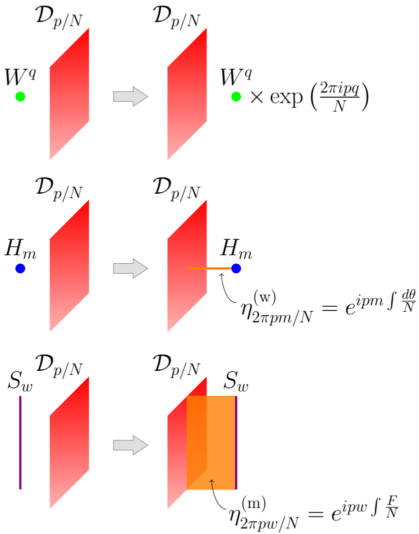
Symm. Op.	**Charged Op.**	**Degree**	**Invertible?**
$\eta_\alpha^{(\mathrm{m})}$	H_m	1-form symm.	Yes
$\eta_\alpha^{(\mathrm{w})}$	S_w	2-form symm.	Yes
$\mathcal{D}_{p/N}$	**?**	1-form symm.	**No**

- How does $\mathcal{D}_{p/N}$ act on extended operators?

Action of $\mathcal{D}_{p/N}$

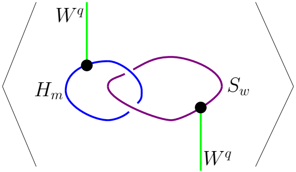


	x^1	x^2	x^3	x^4
$\mathcal{D}_{p/N}$	×	×		
$W^q \equiv e^{iq\oint A}$				×
H_m				×
S_w	×			×
$\eta^{(\mathrm{w})}_{2\pi p/N}$			×	
$\eta^{(\mathrm{m})}_{2\pi p/N}$	×		×	



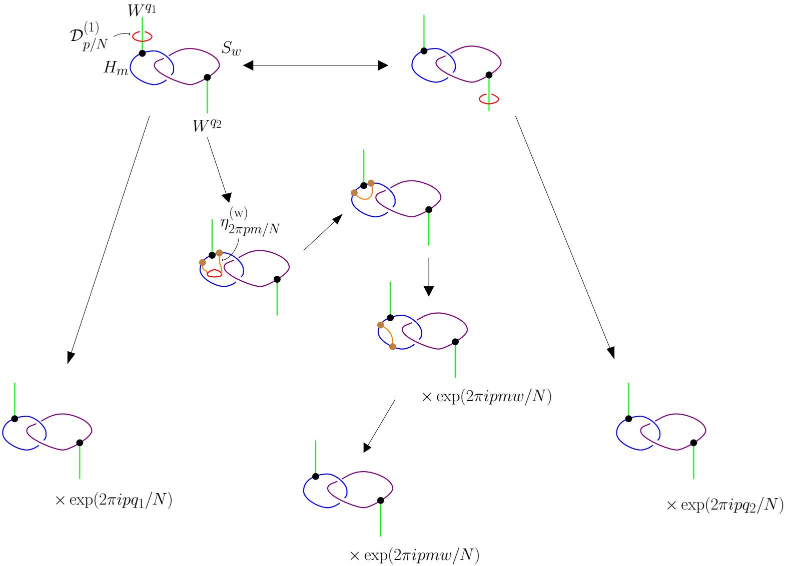
Selection Rules and Witten Effect

- The topological operator $\mathcal{D}_{p/N}$ defines a noninvertible 1-form symmetry.
- Just like ordinary symmetries, it imposes selection rules on correlation functions. For instance, [related to "charge teleportation" in Fukuda-Yonekura 2020]

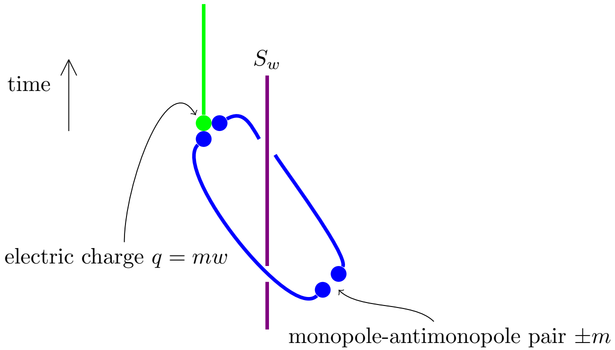


$$= \quad 0 \quad \text{unless} \quad q = mw$$

Selection Rules and Witten Effect



Selection Rules and Witten Effect



- These selection rules are compatible with the Witten effect.
- In a sense, Witten effect (in the presence of a dynamical axion) is reinterpreted as an exact selection rule coming from a noninvertible global symmetry.

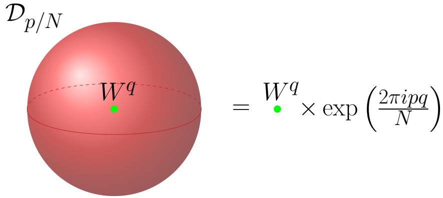
Noninvertible Gauss Law

Noninvertible Gauss Law

$$\mathcal{D}_{p/N} \equiv \int [D\phi][Dc] \exp\left[\oint\left(\frac{2\pi p}{e^2 N}\star F + \frac{iN}{2\pi}\phi dc + \frac{ip}{2\pi}\theta dc + \frac{i}{2\pi}\phi dA\right)\right]$$

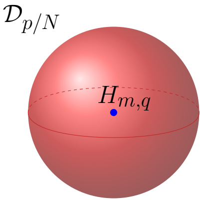
- Now, let us try to use the new topological surface operator $\mathcal{D}_{p/N}$ to measure electric charges as in the usual Gauss law!
- On **Wilson lines**, $\mathcal{D}_{p/N}$ acts in the same way as the ordinary electric 1-form symmetry:

  [Euler counterterm is adjusted such that expectation value of $\mathcal{D}_{p/N}$ on S^2 is equal to 1.]



Noninvertible Gauss Law

- For the dyonic line $H_{m,q} \equiv H_m W^q$, we have:

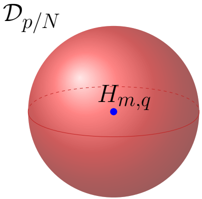


$$= \begin{cases} 0 & \text{if } m \neq 0 \bmod N \\ H_{m,q} \times \exp\left(\frac{2\pi i p q}{N}\right) & \text{if } m = 0 \bmod N \end{cases}$$

- By using different coprime pairs of p and N, we can measure $q \in \mathbb{Z}_m$, but not any more than that.

- Note that the "0" on RHS is a multiplicative 0. It does not mean charge 0, instead it means that the correlation function vanishes.

Noninvertible Gauss Law

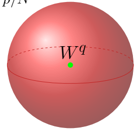


$$= \begin{cases} 0 & \text{if } m \neq 0 \bmod N \\ H_{m,q} \times \exp\left(\frac{2\pi i p q}{N}\right) & \text{if } m = 0 \bmod N \end{cases}$$

- This is indeed consistent with the expectation from the Witten effect. Under $\theta \to \theta + 2\pi$, $q \to q + m$.
- Thus, in the presence of the dynamical axion θ, the electric charge q of a dyon $H_{m,q}$ is well-defined only modulo m. The $\mathcal{D}_{p/N}$ operator precisely measures this value $q \in \mathbb{Z}_m$.

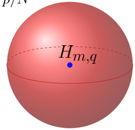
Noninvertible Gauss Law

$\mathcal{D}_{p/N}$



$= W^q \times \exp\left(\frac{2\pi i p q}{N}\right)$

$\mathcal{D}_{p/N}$



$$= \begin{cases} 0 & \text{if } m \neq 0 \bmod N \\ H_{m,q} \times \exp\left(\frac{2\pi i p q}{N}\right) & \text{if } m = 0 \bmod N \end{cases}$$

- Intuitively speaking, this "noninvertible Gauss surface" $\mathcal{D}_{p/N}$ always does its best to measure the electric charge of the surrounded particle, and gives us the best sensible answer.
- However, when it fails to assign any unambigous (i.e. gauge-invariant) value for the electric charge, it simply spits out 0! [c.f. Córdova-Ohmori 2022; Chen-Tanizaki 2022]
- We call this the Noninvertible Gauss Law in the axion-Maxwell theory.

No Global Symmetry Conjecture

Noninvertible Symmetry and Quantum Gravity

- One can also make a connection to various conjectures in **quantum gravity**.
- In quantum gravity, there are two pieces of lore:

No Global Symmetry Conjecture:

There is no global symmetry in quantum gravity.

Completeness Hypothesis:

The spectrum of gauge charges must be complete.

- The two statements are often related, and in some cases, equivalent. For instance, in $U(1)$ gauge theory **without axion**, completeness is **equivalent** to no electric 1-form symmetry.

Noninvertible Symmetry and Quantum Gravity

No invertible global symmetry $\not\Longleftrightarrow$ Completeness of spectrum

No invertible **and** non-invertible global symmetry $\Longleftrightarrow$ Completeness of spectrum

- However, when there is an axion, the equivalence between completeness and no invertible electric 1-form symmetry breaks down. [c.f. Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021]
- The equivalence is **restored** if we include the non-invertible electric 1-form symmetry $\mathcal{D}_{p/N}$! [**YC**-Lam-Shao 2022]
- Generally, it is argued that the no invertible **and** non-invertible global symmetry is equivalent to completeness. [Shao-Rudelius 2020; Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021]

Summary and Outlook

- In 4d axion-Maxwell theory, there exists a noninvertible 1-form generalized global symmetry.
- The associated noninvertible Gauss law measures the probe electric charges to the extent allowed by the ambiguity coming from the Witten effect.
- It would be interesting to study phenomenological implications on more realistic axion models.
- Similar construction applies to many supergravity theories [García-Valdecasas 2023]. It might be interesting to study similar noninvertible Gauss laws for various branes there.

Thank you!