Piano Tuning

Piano ✵ Tuning

For Physicists & Engineers

using your

Laptop, Microphone, and Hammer

by

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at

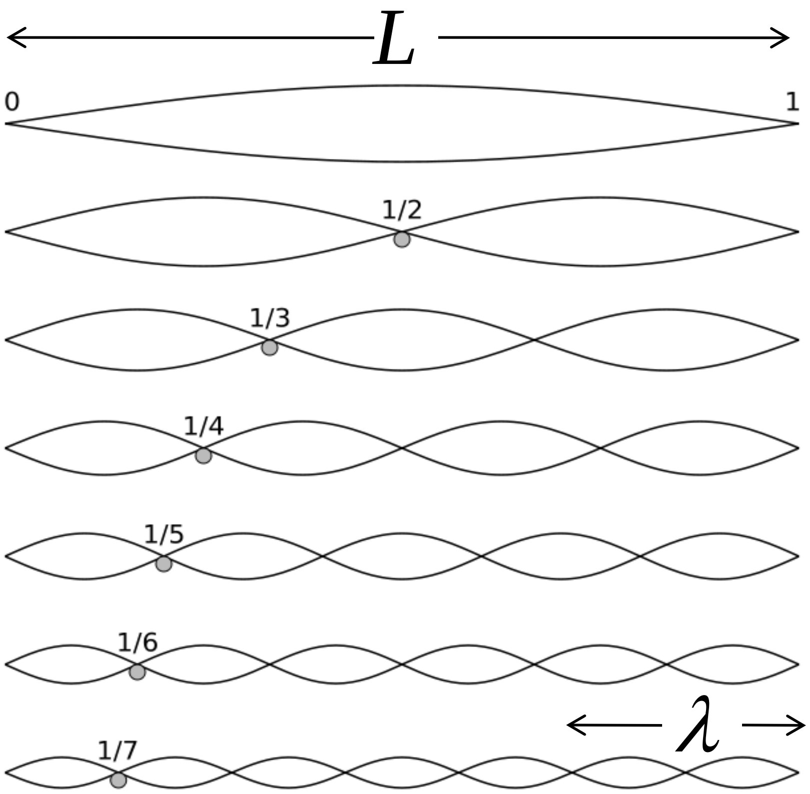
3:00 pm Room 130 Hahn North
March 17, 2012

What our $50 piano sounded like when delivered.

So far: cleaned, fixed four keys, raised pitch a half-step to set A4 at 440 Hz, and did a rough tuning…

Bravely put your ‘VT physics education’ to work on that ancient piano!

Tune: to what? why? how?
Regulate: what?
Fix keys: how?



A piano string is fixed at its two ends, and can vibrate in several harmonic modes.

$$L = n\frac{\lambda}{2};$$

$$f_n = \frac{v}{\lambda} = n\frac{v}{2L} = nf_0$$

$$\omega_n = 2\pi f_n$$

frequency of string = frequency of sound

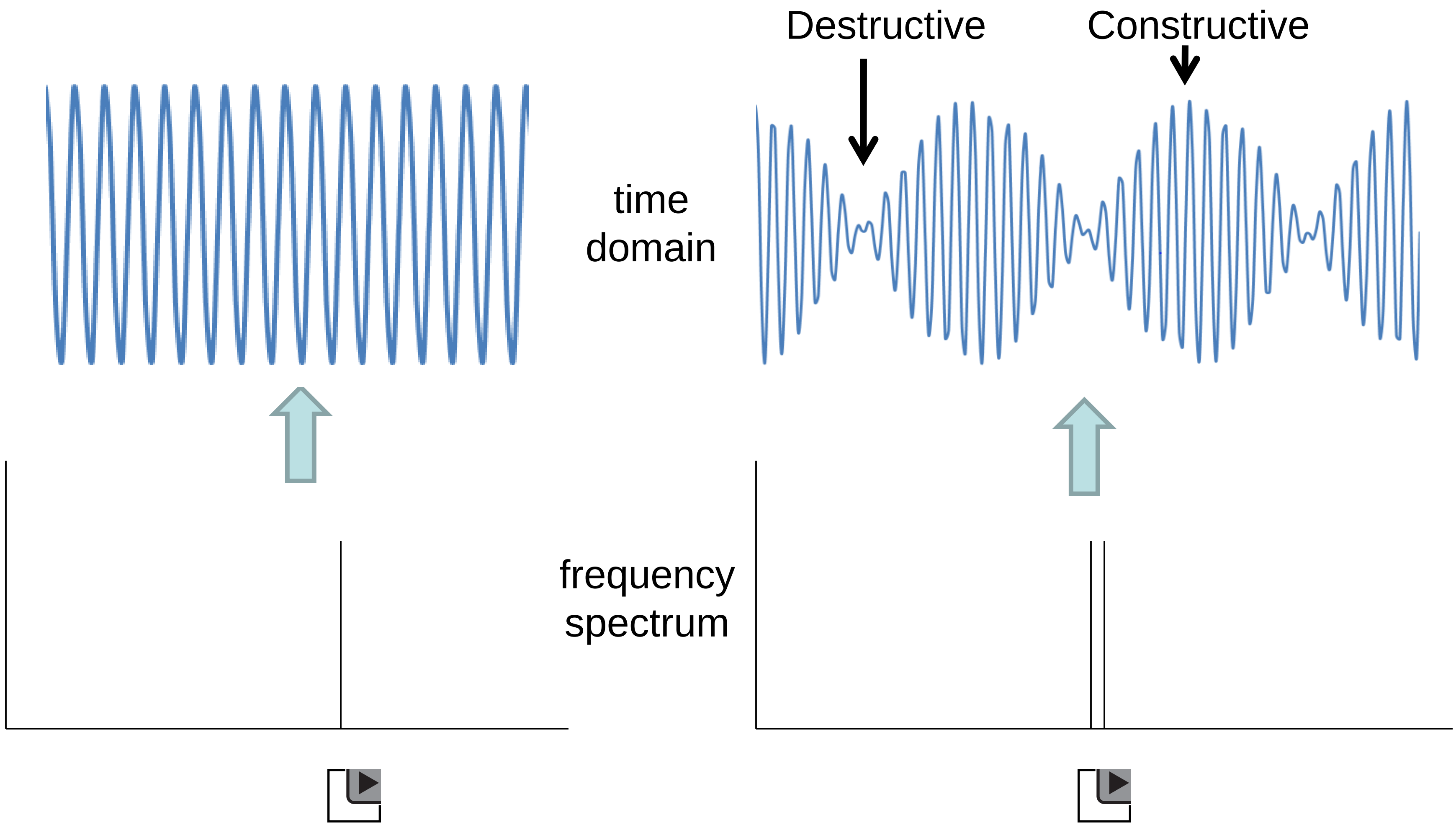
(λ of string ≠ λ of sound)

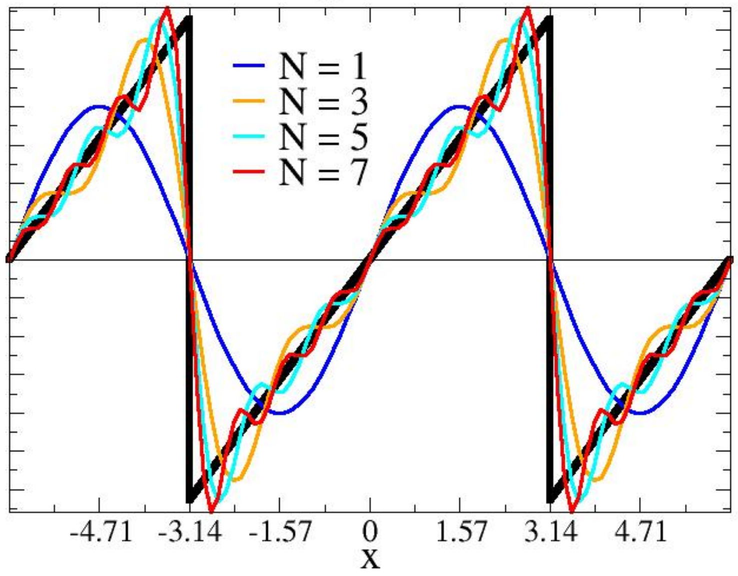
[v = speed of wave on string]

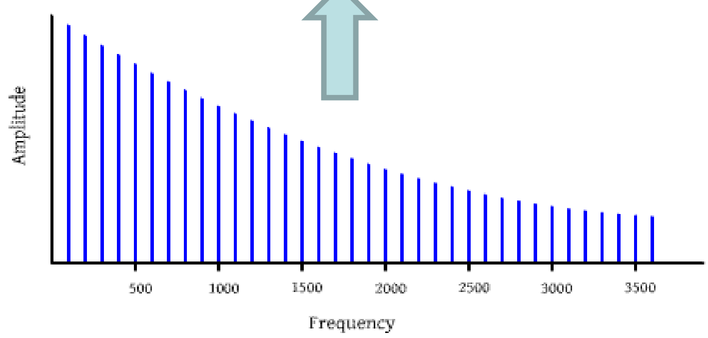
"Pluck" center → mostly 'fundamental'
"Pluck" near edge → many higher 'harmonics'

What you hear is the sum (transferred into air pressure waves).

$$P(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) + a_3 \sin(\omega_3 t) + \ldots$$


Destructive
Constructive
time
domain
frequency
spectrum





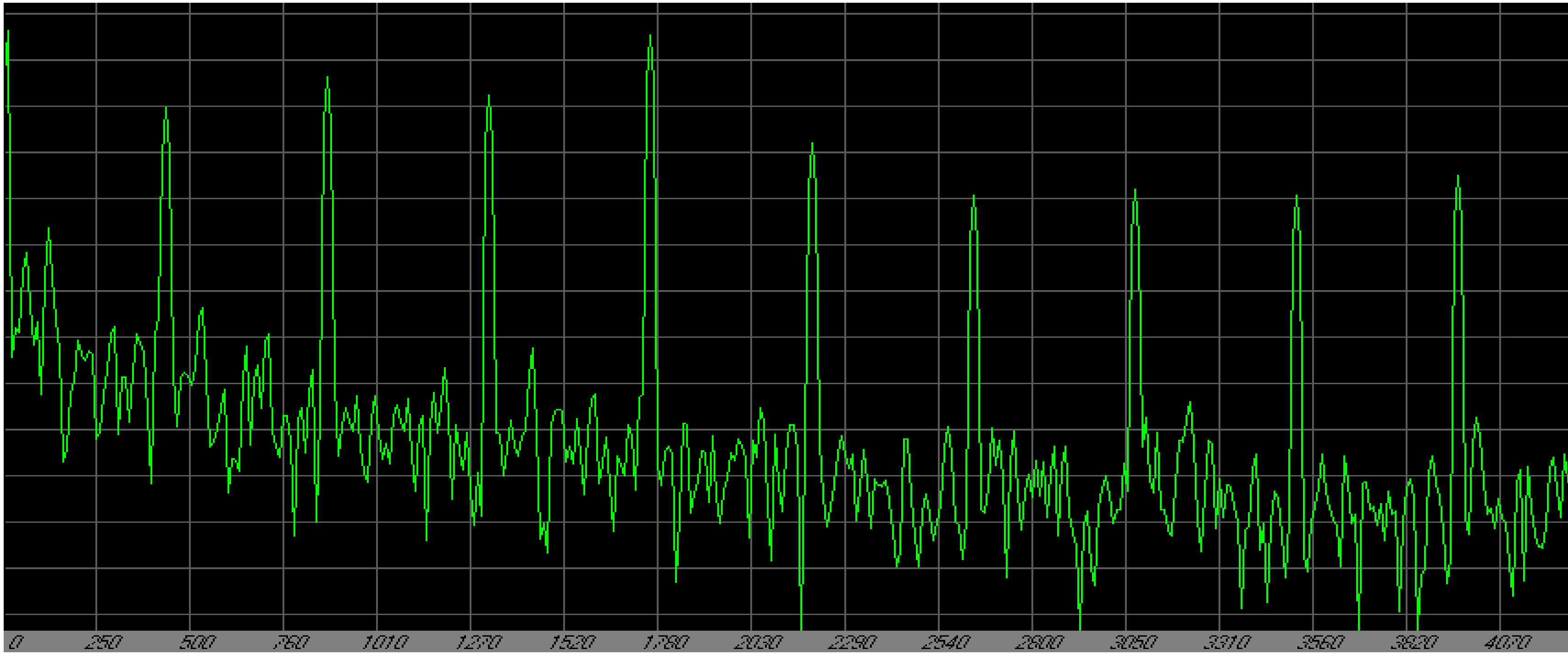
frequency content determines ‘timbre’

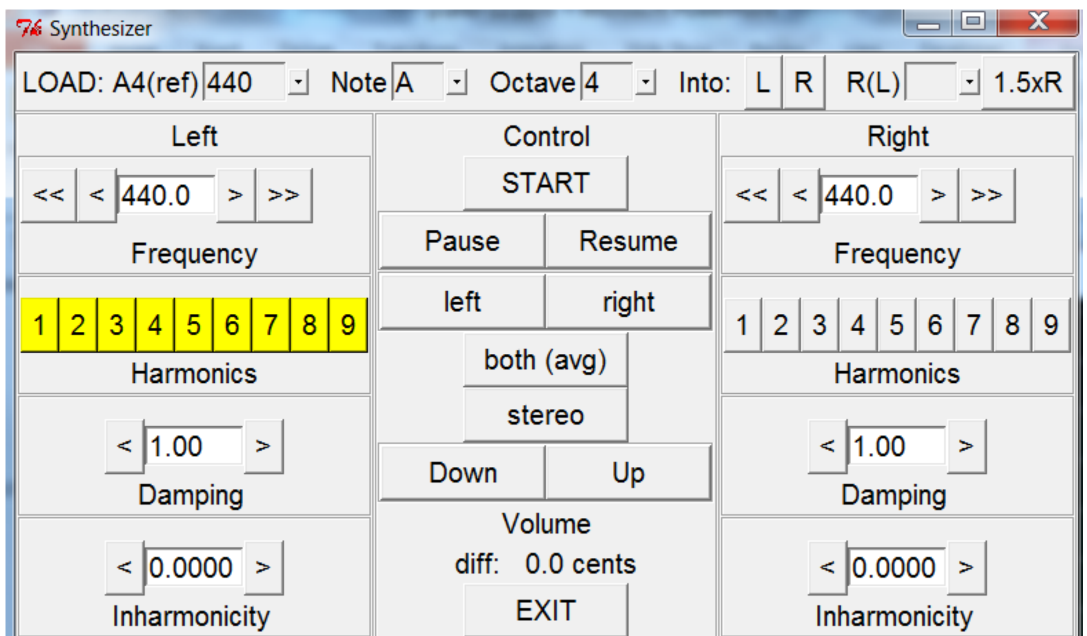
Given only the ‘sum’,

what were the components?

Fourier Analysis

“How much of the sum comes from individual components”


0
250
500
760
1010
1270
1520
1780
2030
2290
2540
2800
3050
3310
3560
3820
4070

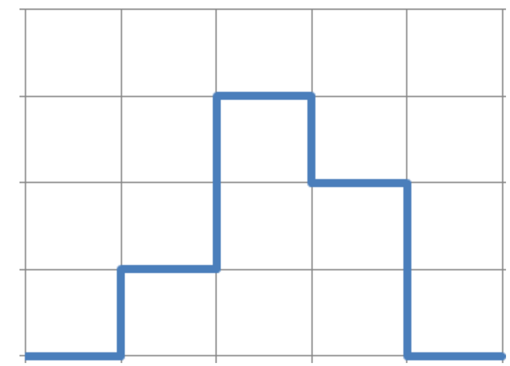

Synthesizer
LOAD: A4(ref) 440
Note A
Octave 4
Into: L R
R(L)
1.5xR
Left
<< < 440.0 > >>
Frequency
1 2 3 4 5 6 7 8 9
Harmonics
< 1.00 >
Damping
< 0.0000 >
Inharmonicity
Control
START
Pause
Resume
left
right
both (avg)
stereo
Down
Up
Volume
diff: 0.0 cents
EXIT
Right
<< < 440.0 > >>
Frequency
1 2 3 4 5 6 7 8 9
Harmonics
< 1.00 >
Damping
< 0.0000 >
Inharmonicity

13 slides on how this is done (just can't resist)

Piano Tuning

Consider a class grade distribution:

P(x):



P(x) **is the number of students versus grade**

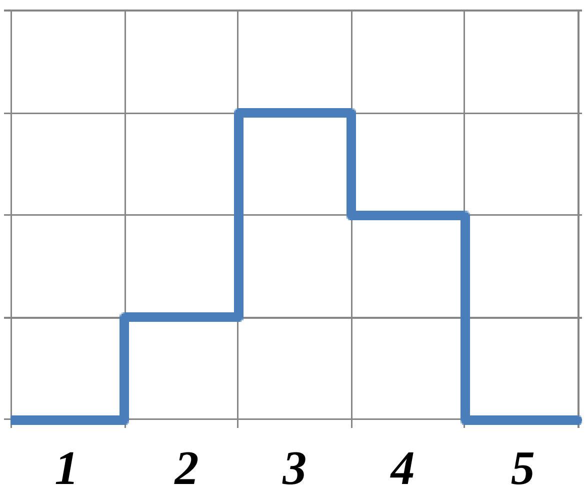
f(x):

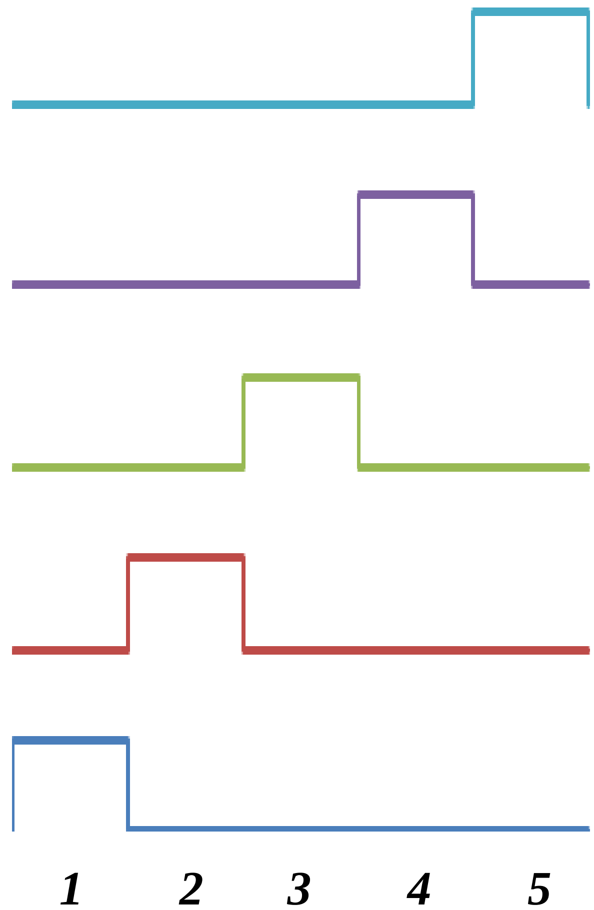
f(x) **is a 1x1 block at a certain grade**

Summing the product of P(x)f(x) gives the number of students with that grade

P(x)f(x):

P(x)	f(x)	∫ P(x)f(x)
"sum"	*"components"*	





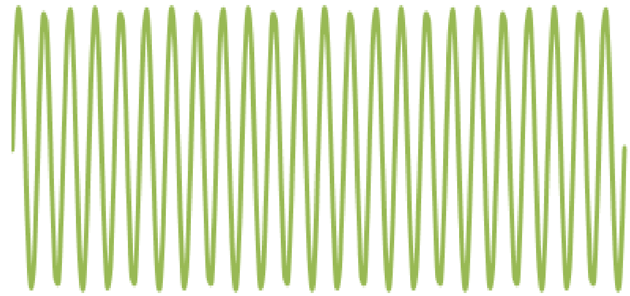
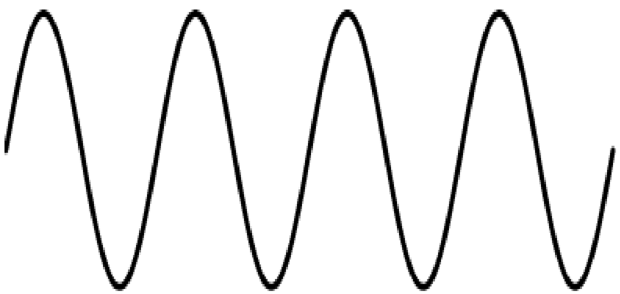
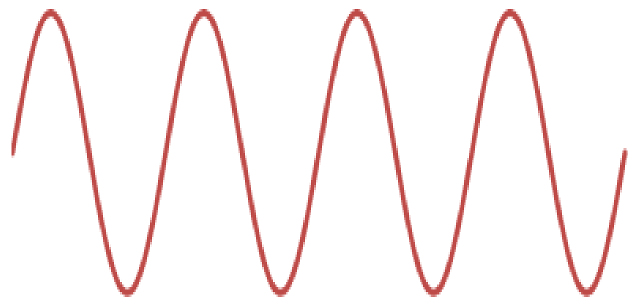
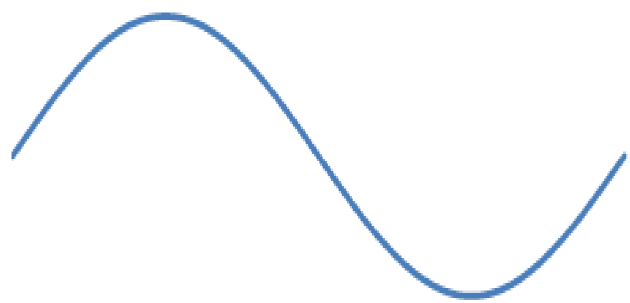
0

2

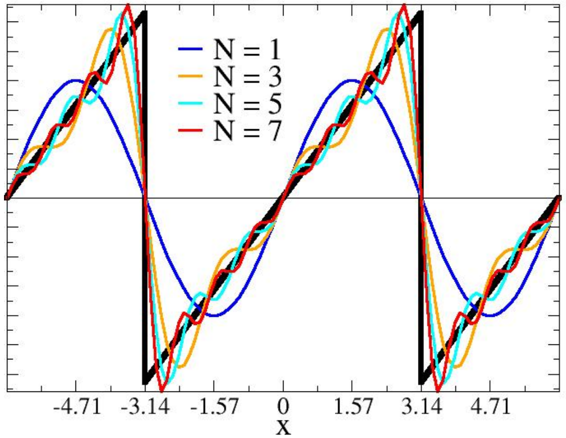
3

1

0

P(t)	f(t)	$\int P(t)f(t)$
		0
		1
		0

An arbitrary waveform can be described by a sum of cosine and sine functions:

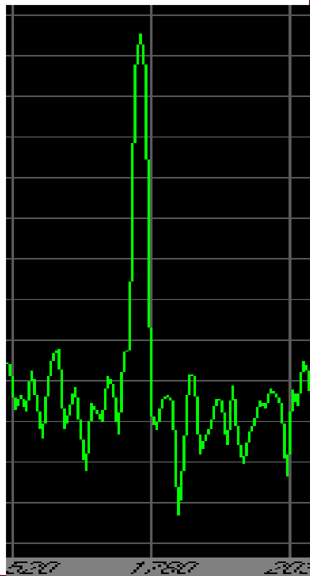


$$P(t) = \sum_{n=0}^{\infty} \left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \right]$$

piano 'note' is a sum of harmonics

want graph of amplitude-vs-frequency

$$\sqrt{a_n^2 + b_n^2} \qquad \omega = 2\pi f$$



finding a_m

$$\int_{cycle} P(t)\cos(\omega_m t)\,dt = \int_{cycle} \sum_{n=0}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \cos(\omega_m t) dt$$

$$\int_{cycle} P(t)\cos(\omega_m t)dt = \pi a_m$$

all terms on right integrate to zero except m^{th} !

$$\int_{cycle} \cos(\omega_n t)\cos(\omega_m t)dt = \pi\delta_{nm} \qquad (0;\text{ or } \pi \text{ if } m = n)$$

$$\int_{cycle} \sin(\omega_n t)\cos(\omega_m t)dt = 0$$

find b_m using $\sin(\omega_m t)$

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typical extraction of properties from a distribution

Piano Tuning

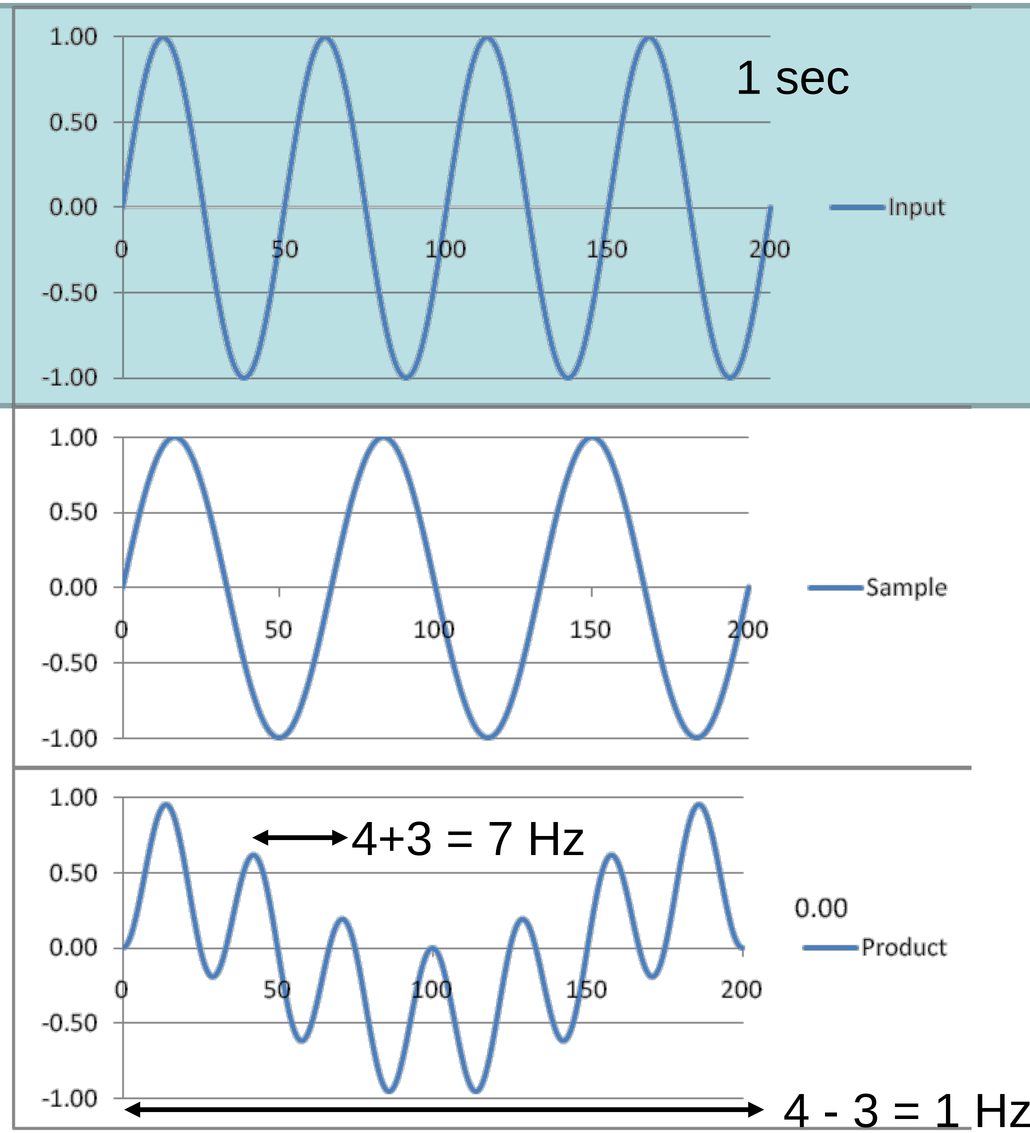
Weighted average	Typical Application (assume P and ψ are normalized)
$f_{avg} = \sum P f$	class grade average center of mass dipole moments
$f_{avg} = \int P\, f$	(same as above, but for continuous distributions) e.g.: Maxwell Boltzmann velocity distributions
$a_n = \int_{-\pi}^{\pi} P(x) \frac{1}{\pi} \cos(nx) dx$	Fourier component of $P(x) = \sum_0^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$
$f_{avg} = \int f \lvert\psi\rvert^2$	commuting quantum mechanical variables
$f_{avg} = \langle \psi \lvert f \rvert \psi \rangle$	non-commuting quantum mechanical variables
$rate \propto \lvert \langle \psi_f \lvert f \rvert \psi_i \rangle \rvert^2$	Fermi’s golden rule for transitions between two states.

200 Samples, every 1/200 second, giving f_0 = 1 Hz

Input 4Hz pure sine wave

Look for 3Hz component

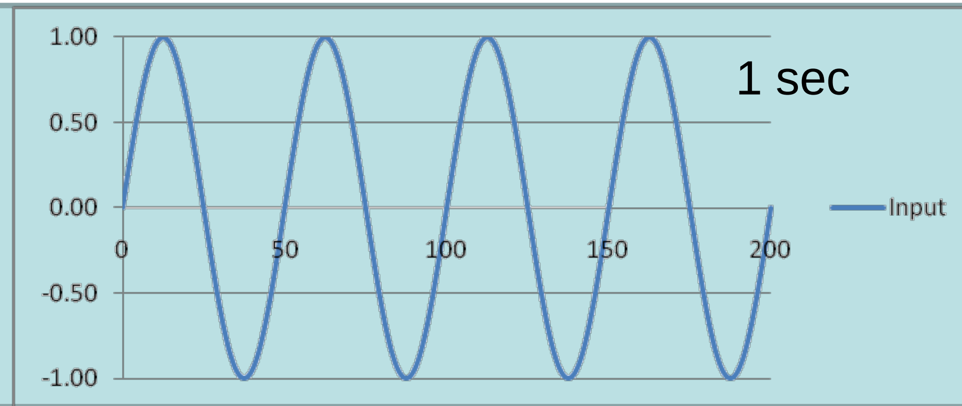
Multiply



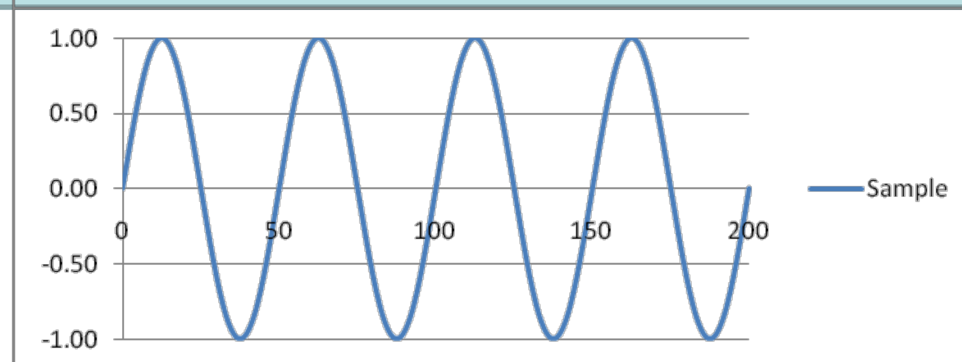
Average $\sin(4t)\sin(3t) = \frac{1}{2}[\cos(1t) - \cos(7t)]$ **AVG = 0**

Piano Tuning

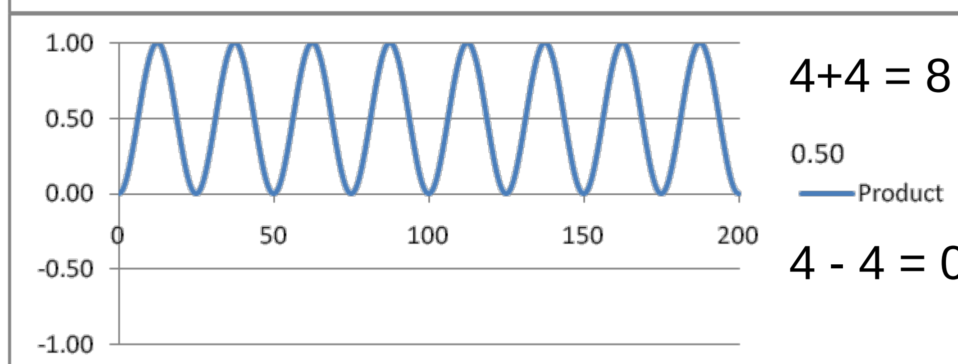
Input 4Hz pure sine wave



Look for 4Hz component



Multiply



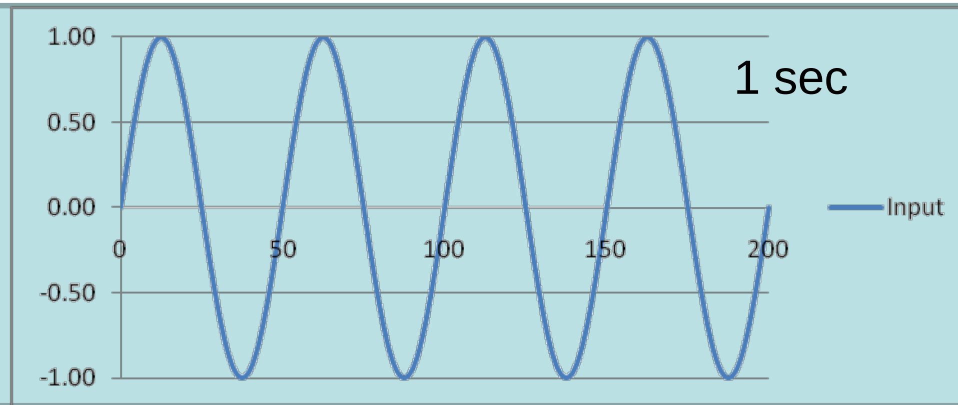
4+4 = 8 Hz

4 - 4 = 0 Hz

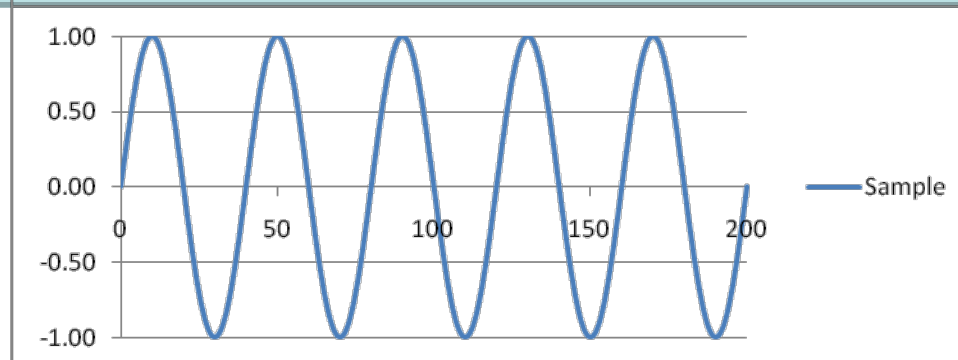
Average $\sin(4t)\sin(4t) = \frac{1}{2}[\cos(0t) - \cos(8t)]$ **AVG = 1/2**

Piano Tuning

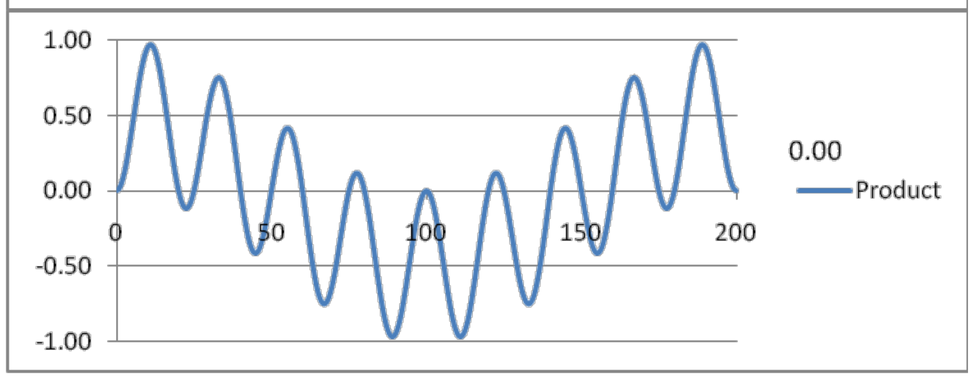
Input 4Hz pure sine wave



Look for 5Hz component



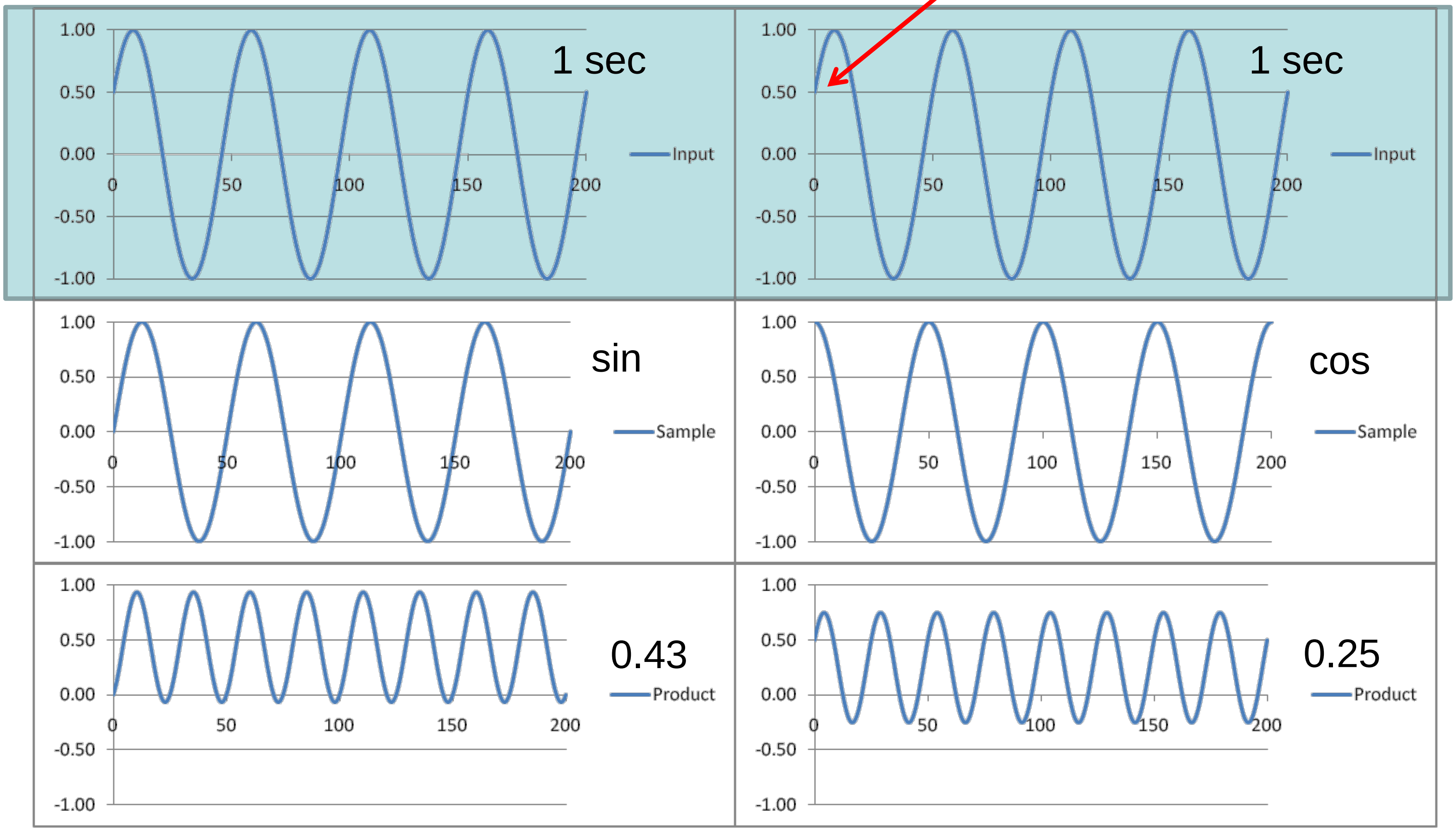
Multiply



Average $\sin(4t)\sin(5t) = \frac{1}{2}[\cos(1t) - \cos(9t)]$ **AVG = 0**

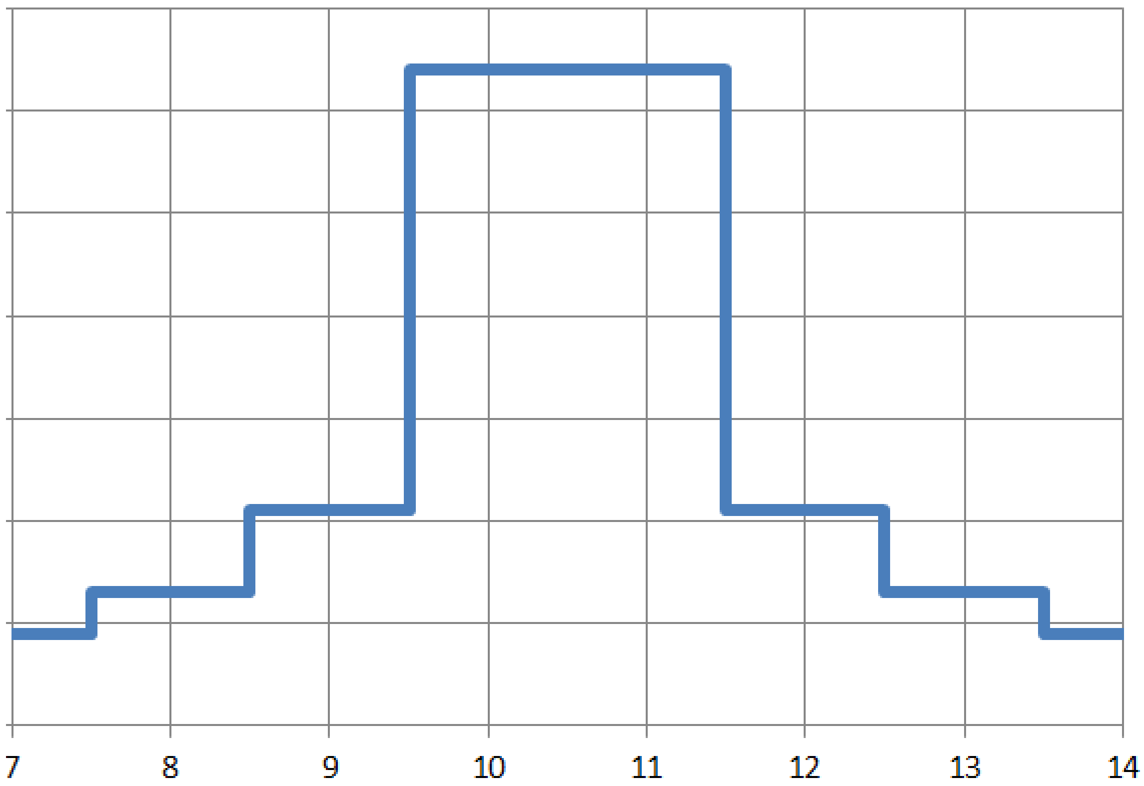
Great, picked out the 4 Hz input. But what if the input phase is different?

Use COS as well. For example: 4Hz, $\phi_0 = 30^o$; sample 4 Hz



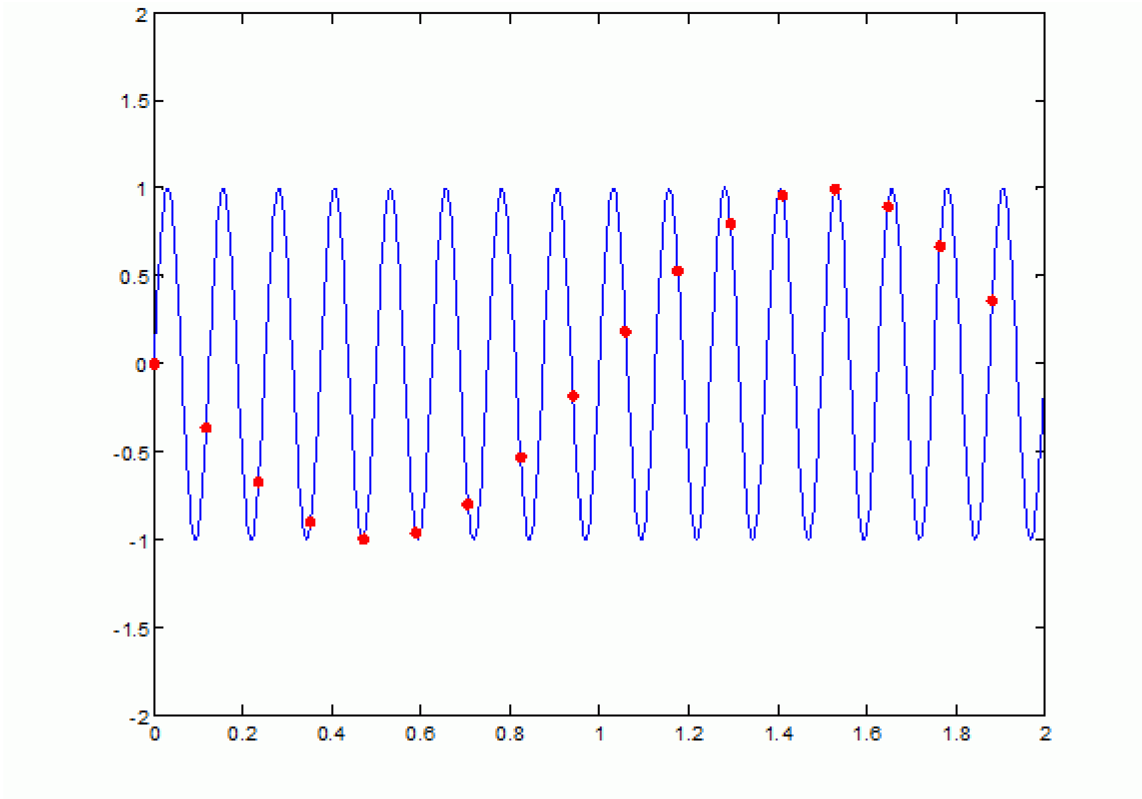
$(0.43^2 + 0.25^2)^{1/2} = 1/2$ Right On!

Signal phase does not matter. What about input at 10.5 Hz?



Finite Resolution

Remember, we only had 200 samples, so there is a limit to how high a frequency we can extract.   Consider 188 Hz, sampled every 1/200 seconds:



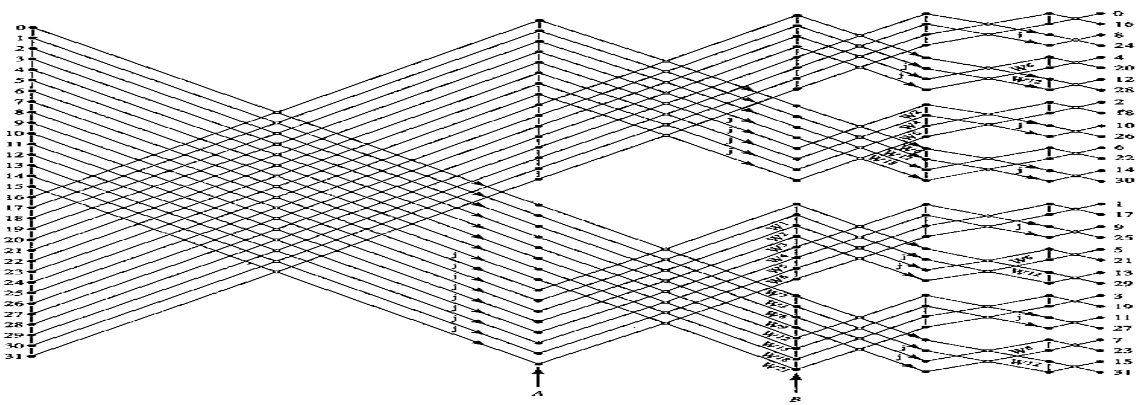
Nyquist Limit

Sample > 2x frequency of interest;

lots of multiplication & summing → slow…

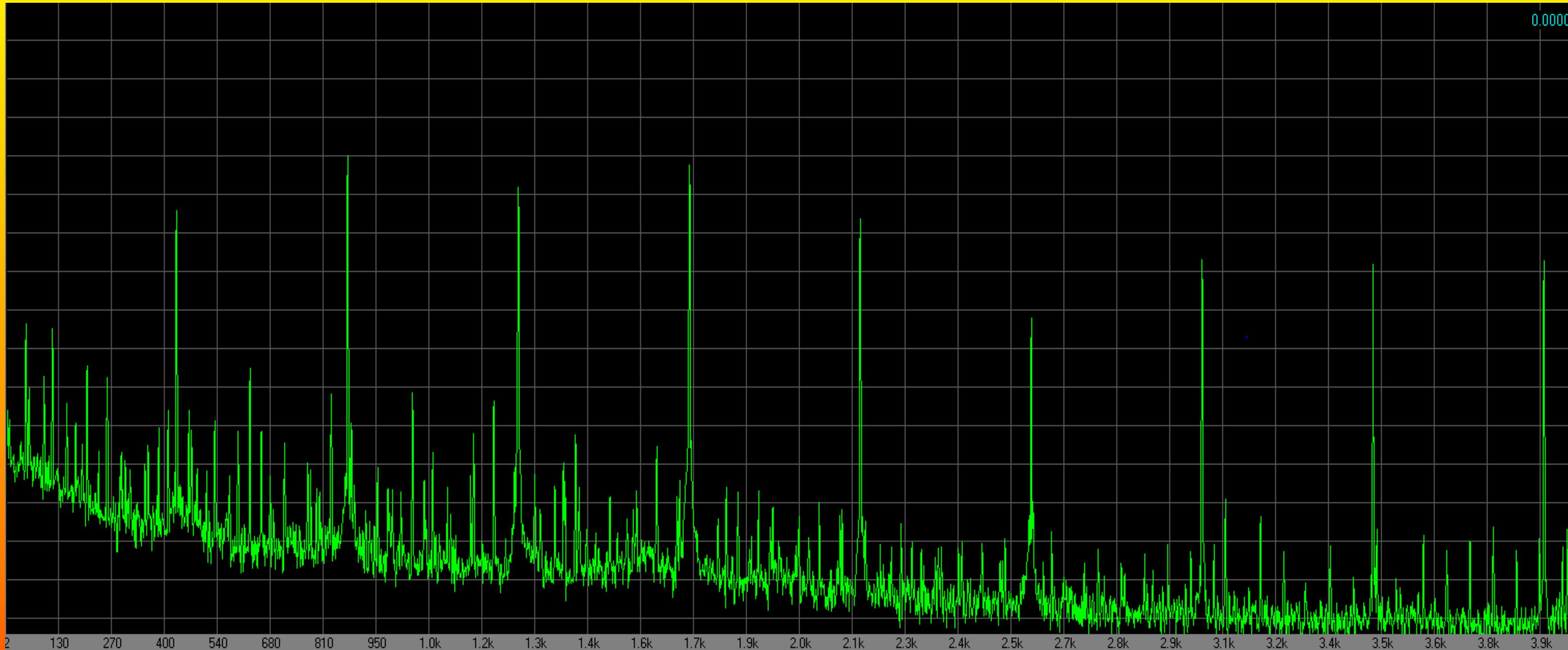
Fast Fourier Transforms

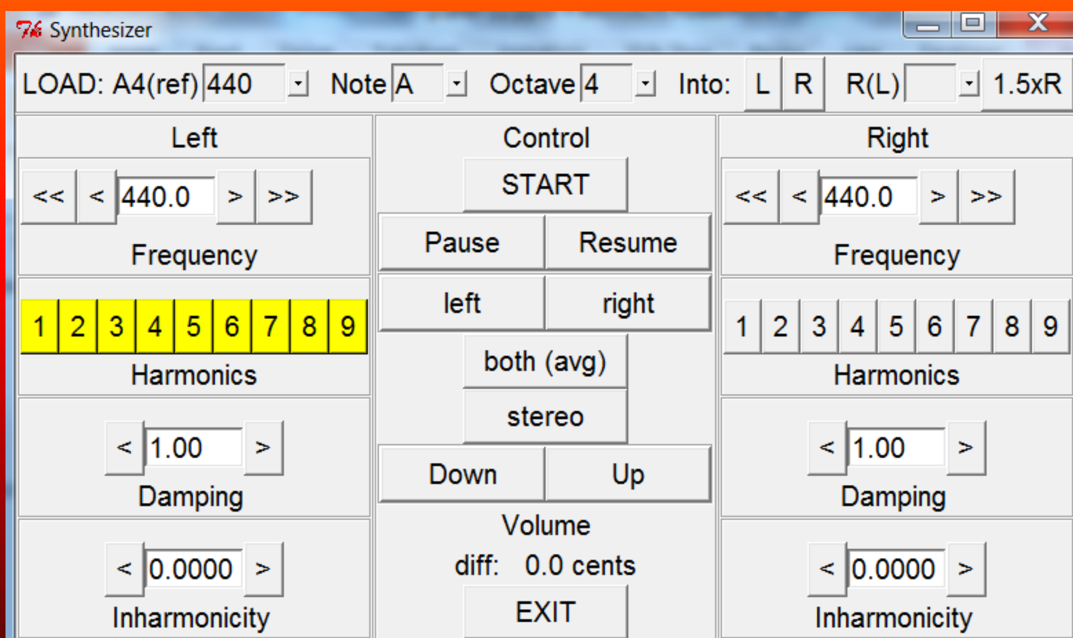
- uses Euler's $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- several very clever features
- $\Rightarrow$ 1000's of times faster



Free FFT Spectrum Analyzer:
http://www.sillanumsoft.org/download.htm
"Visual Analyzer"

Piano Tuning

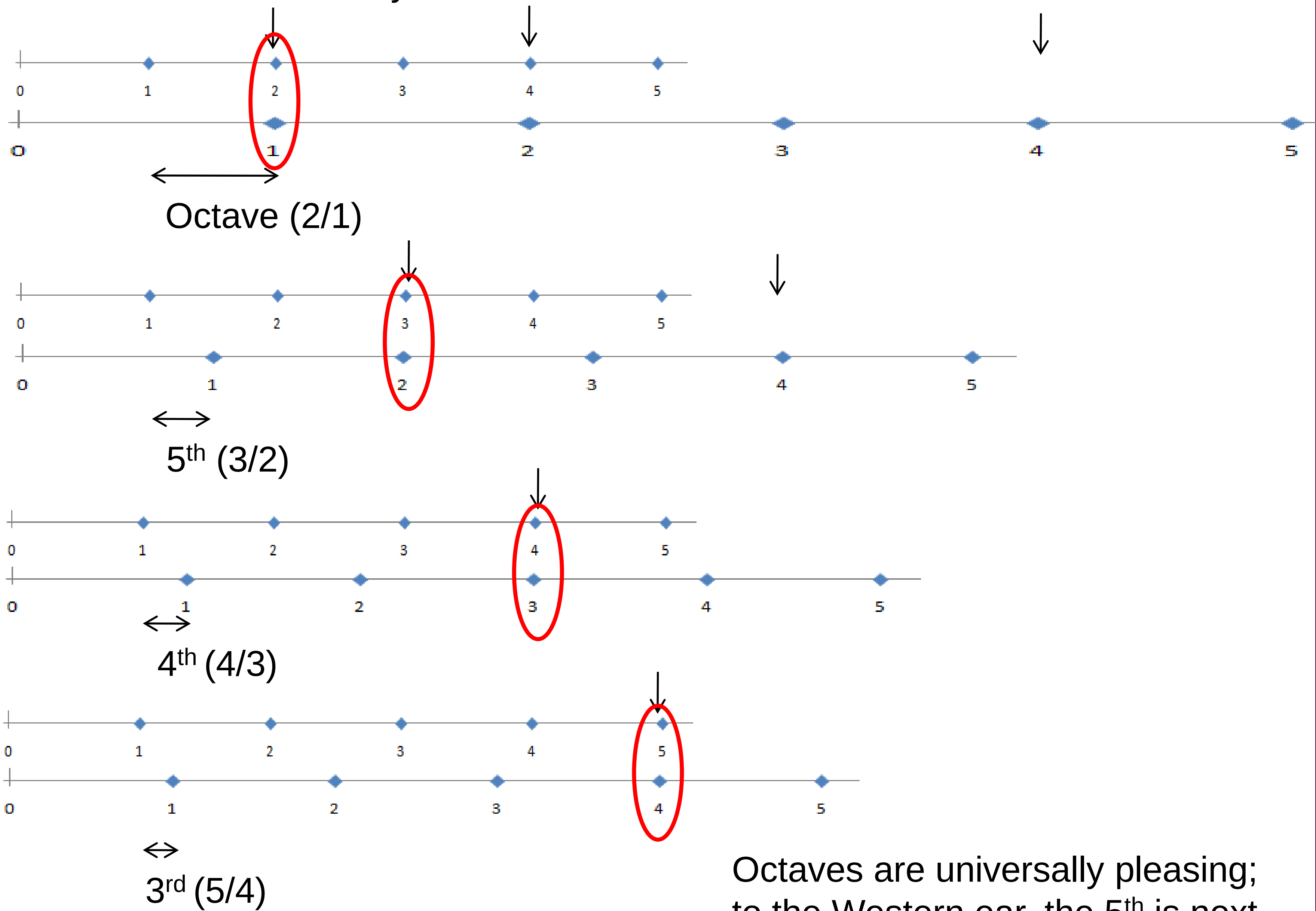




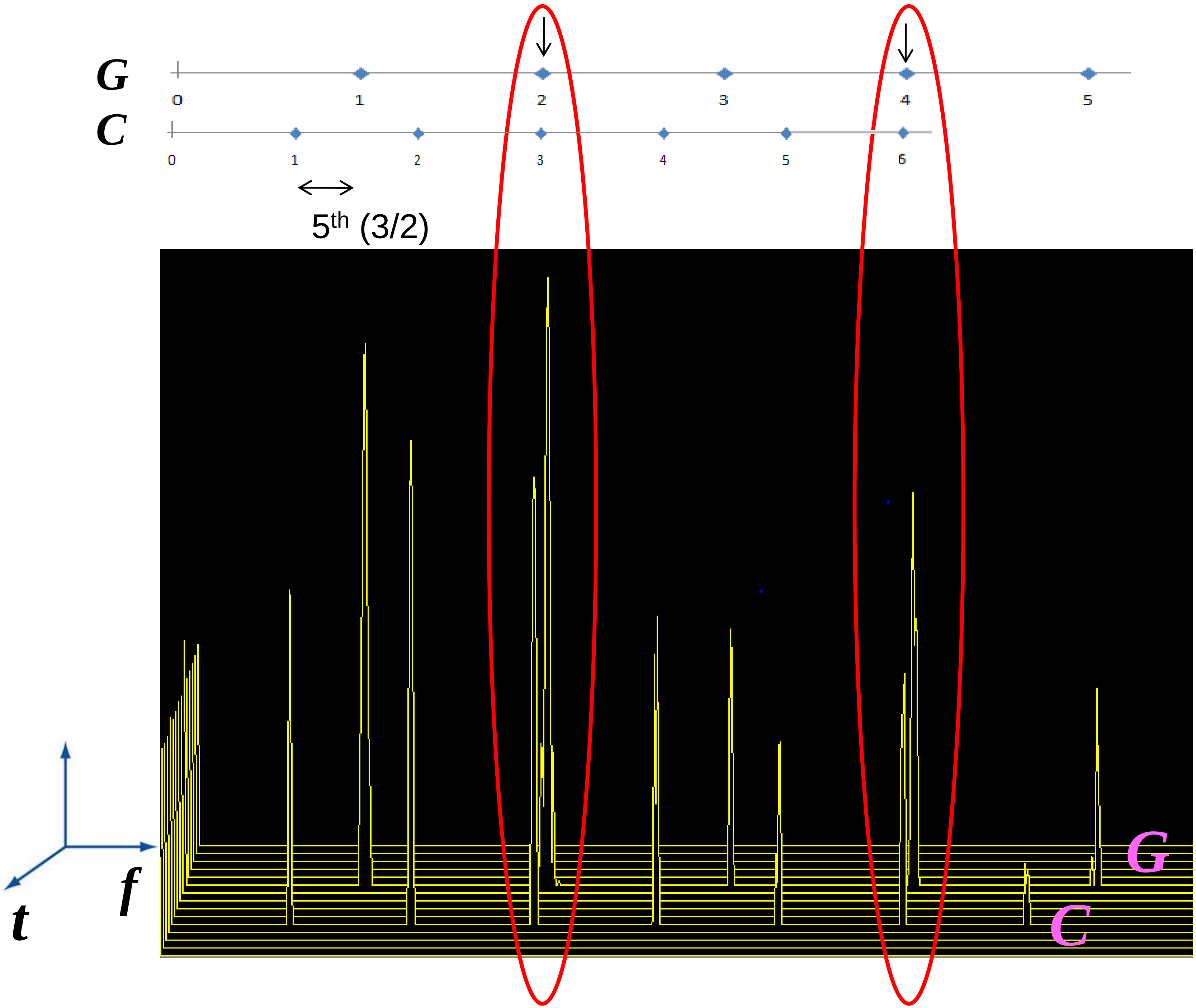
40960 sample/s
32768 samples

= 1.25 Hz resolution

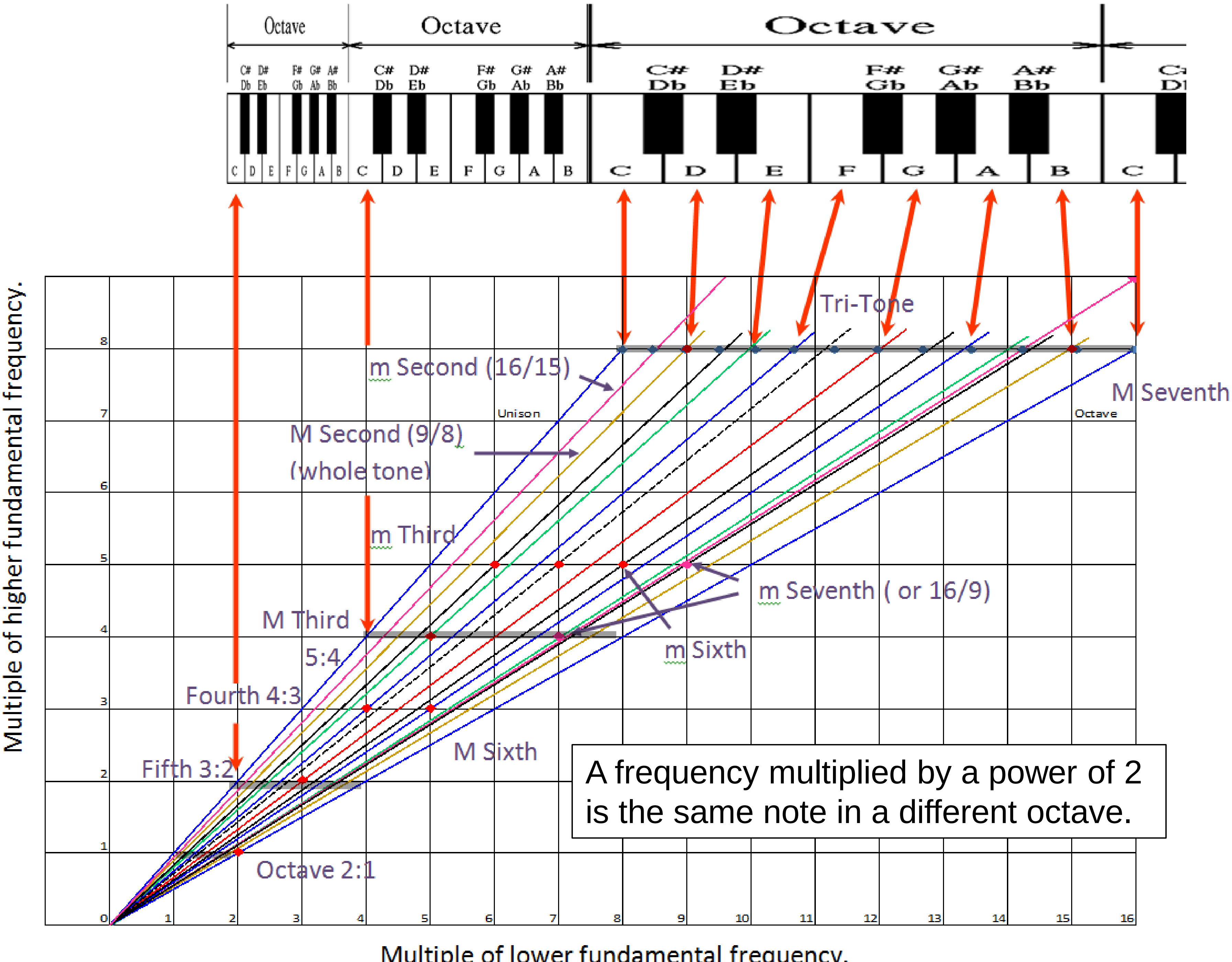
Why some notes sound ‘harmonious’



Octaves are universally pleasing; to the Western ear, the 5th is next most important.


G
0
1
2
3
4
5
C
0
1
2
3
4
5
6
5th (3/2)
t
f
G
C

Piano Tuning



"Circle of 5th s"

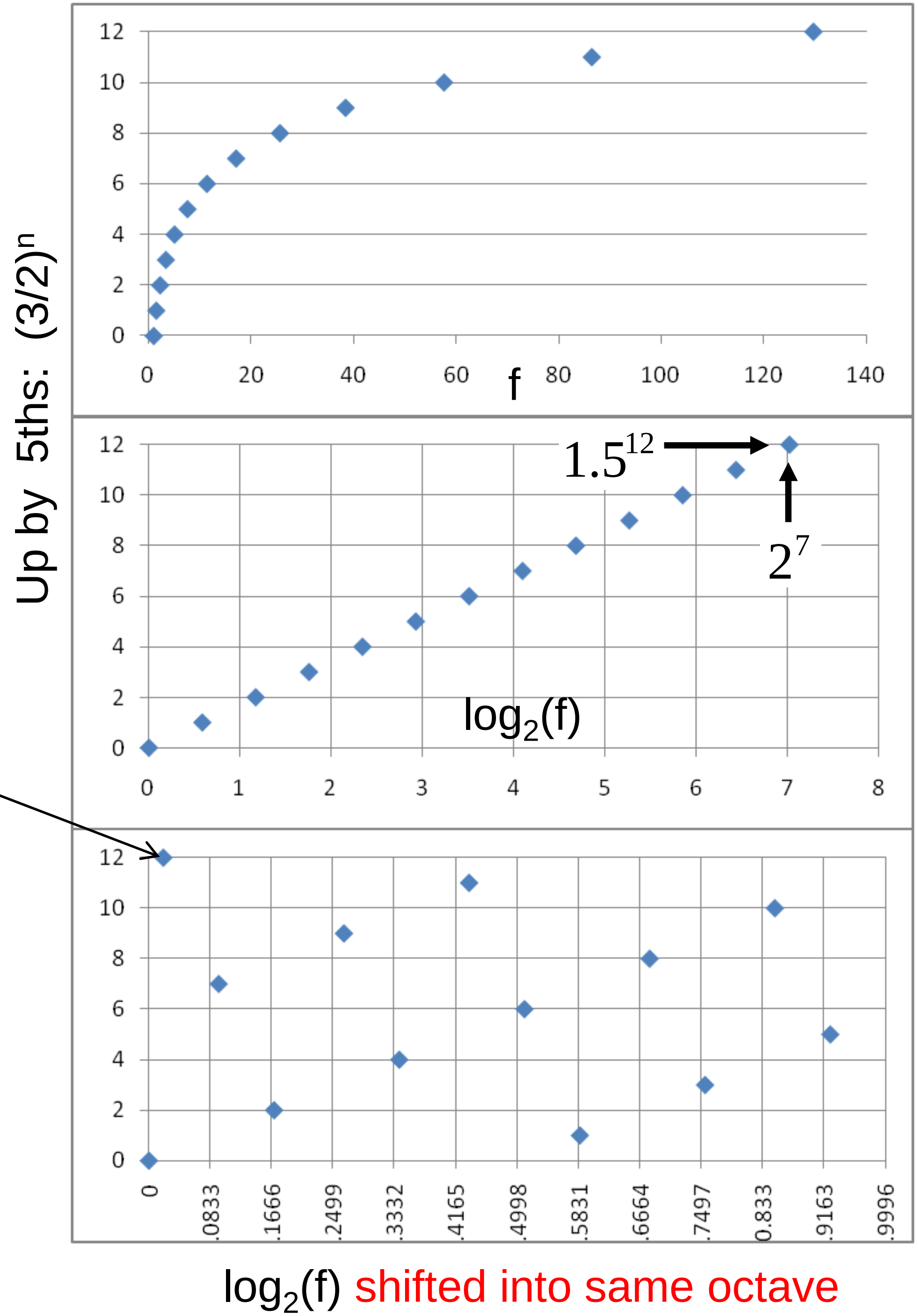
Going up by 5ths
12 times brings you ***very***
near the same note
(but 7 octaves up)

(this suggests perhaps
12 notes per octave)

"Wolf " fifth

We define the number of
'cents' between two notes as
1200 * $\log_2(f_2/f_1)$

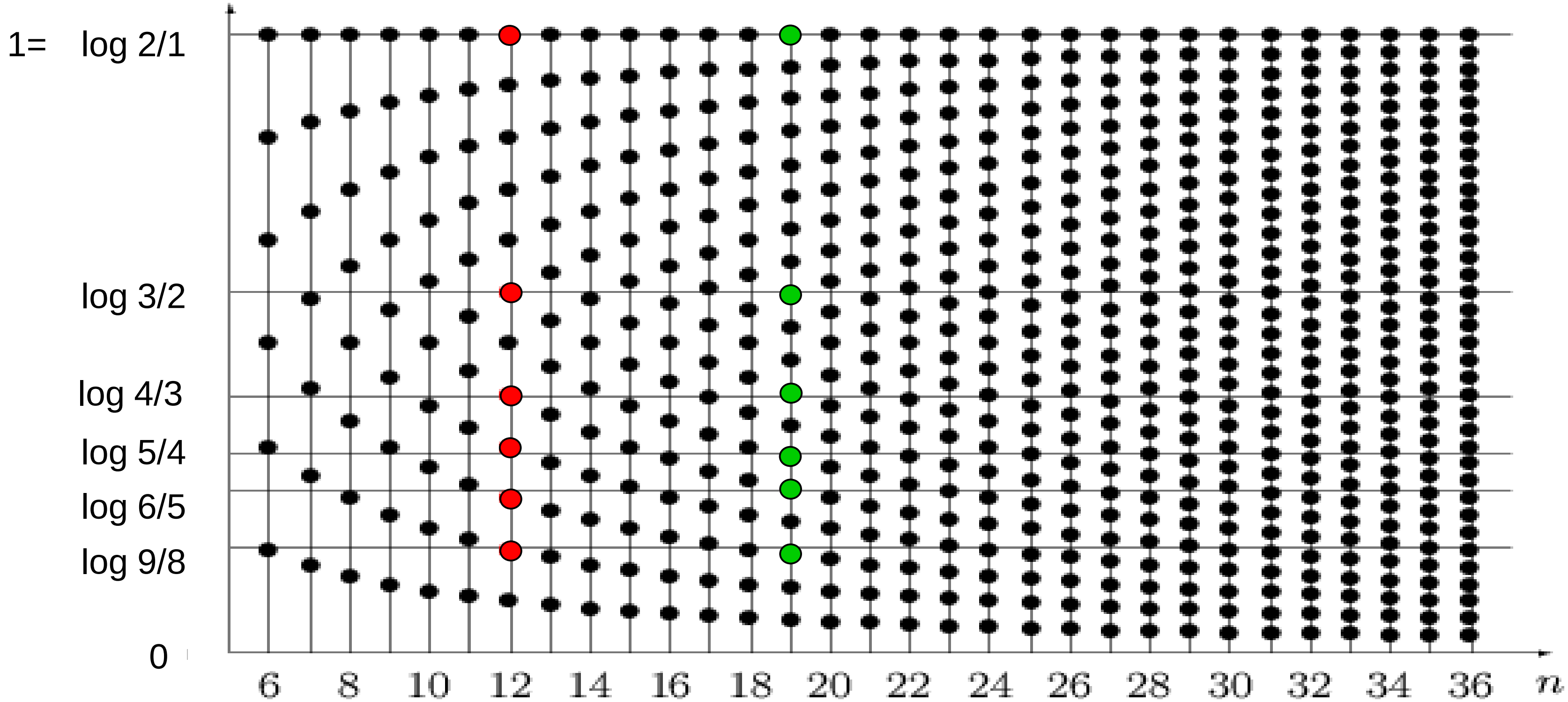
Octave = 1200 cents
"Wolf " fifth off by 23 cents.

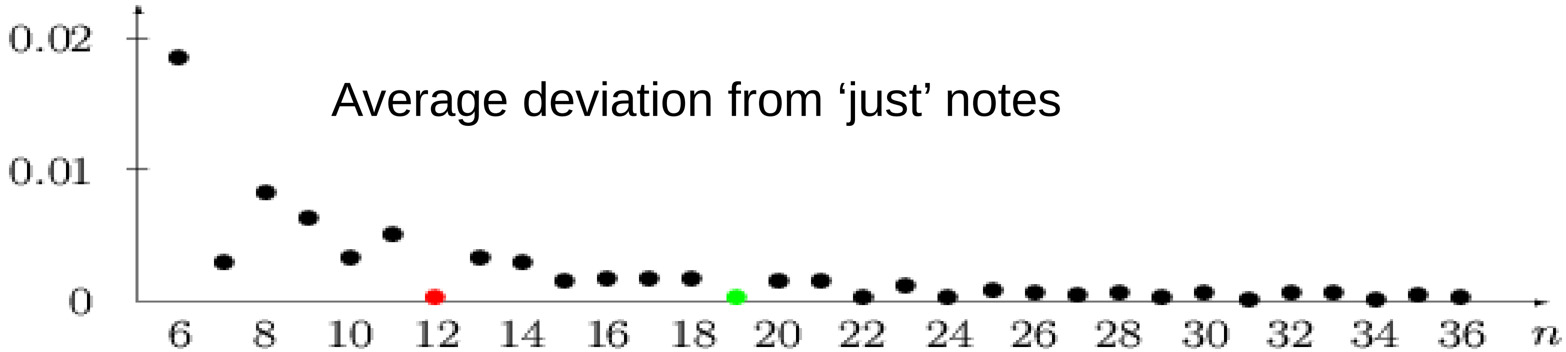


Piano Tuning

Piano Tuning

$\log_2$ of ‘ideal’ ratios   Options for equally spaced notes





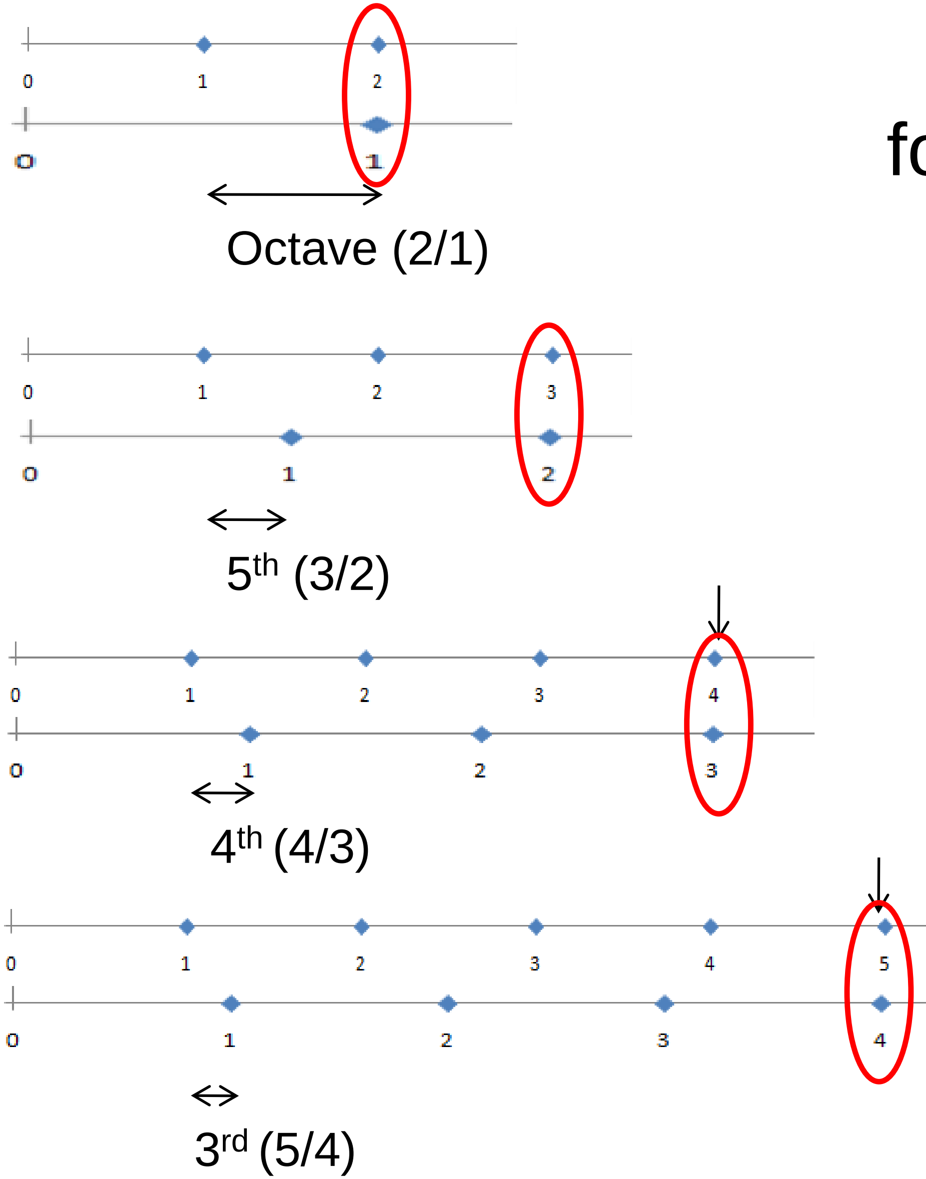
We’ve chosen 12 **EQUAL** tempered steps; could have been 19 just as well…

Piano Tuning

Typically set A4 to 440 Hz

Interval	Equal Temperament Frequency Ratio			Difference	Harmonic Series Frequency Ratio		
Octave	$\left(\sqrt[12]{2}\right)^{12}$	=	2.0000	0.0000	2.0000	=	2/1
Major Seventh	$\left(\sqrt[12]{2}\right)^{11}$	=	1.8877	0.0127	1.8750	=	15/8
Minor Seventh	$\left(\sqrt[12]{2}\right)^{10}$	=	1.7818	0.0318	1.7500	=	7/4
Major Sixth	$\left(\sqrt[12]{2}\right)^{9}$	=	1.6818	0.0151	1.6667	=	5/3
Minor Sixth	$\left(\sqrt[12]{2}\right)^{8}$	=	1.5874	-0.0126	1.6000	=	8/5
Perfect Fifth	$\left(\sqrt[12]{2}\right)^{7}$	=	1.4983	-0.0017	1.5000	=	3/2
Tritone	$\left(\sqrt[12]{2}\right)^{6}$	=	1.4142	0.0000	1.4142	=	$\sqrt{2}/1$
Perfect Fourth	$\left(\sqrt[12]{2}\right)^{5}$	=	1.3348	0.0015	1.3333	=	4/3
Major Third	$\left(\sqrt[12]{2}\right)^{4}$	=	1.2599	0.0099	1.2500	=	5/4
Minor Third	$\left(\sqrt[12]{2}\right)^{3}$	=	1.1892	-0.0108	1.2000	=	6/5
Major Second	$\left(\sqrt[12]{2}\right)^{2}$	=	1.1225	-0.0025	1.1250	=	9/8
Minor Second	$\left(\sqrt[12]{2}\right)^{1}$	=	1.0595	-0.0072	1.0667	=	16/15
Unison	$\left(\sqrt[12]{2}\right)^{0}$	=	1.0000	0.0000	1.0000	=	1/1

What an ‘aural’ tuner does…



for ***equal*** temperament:

tune so that desired harmonics are at the same frequency;

then, set them the required amount off by counting ‘beats’.

Piano Tuning

Equal temperament beatings (all figures in Hz)												
261.626	277.183	293.665	311.127	329.628	349.228	369.994	391.995	415.305	440.000	466.164	493.883	523.251
0.00000			14.1185	20.7648	1.18243		1.77165	16.4810	23.7444			C
		13.3261	19.5994	1.11607		1.67221	15.5560	22.4117			B	
	12.5781	18.4993	1.05343		1.57836	14.6829	21.1538			B♭		
11.8722	17.4610	.994304		1.48977	13.8588	19.9665			A			
16.4810	.938498		1.40616	13.0810	18.8459			A♭				
.885824		1.32724	12.3468	17.7882			G				Fundamental	
	1.25274	11.6539	16.7898			F♯					Octave	
1.18243	10.9998	15.8475			F						Major sixth	
10.3824	14.9580			E							Minor sixth	
14.1185			E♭								Perfect fifth	
		D									Perfect fourth	
	C♯										Major third	
C											Minor third	

From C, set G above it such that an octave and a fifth above the C you hear a 0.89 Hz 'beating'

Interval	Approximate ratio	Beating above the lower pitch	Tempering
Unison	1:1	Unison	Exact
Octave	2:1	Octave	Exact
Major sixth	5:3	Two octaves and major third	Wide
Minor sixth	8:5	Three octaves	Narrow
Perfect fifth	3:2	Octave and fifth	Slightly narrow
Perfect fourth	4:3	Two octaves	Slightly wide
Major third	5:4	Two octaves and major third	Wide
Minor third	6:5	Two octaves and fifth	Narrow

I was hopeless, and even wrote a synthesizer to try and train myself…

but I still couldn't 'hear' it…

These beat frequencies are for the central octave.

Is it hopeless?

not with a little help from math and a laptop…

we (non-musicians) can use a spectrum analyzer…

Piano Tuning

With a (free) “Fourier” spectrum analyzer we can set the pitches exactly!

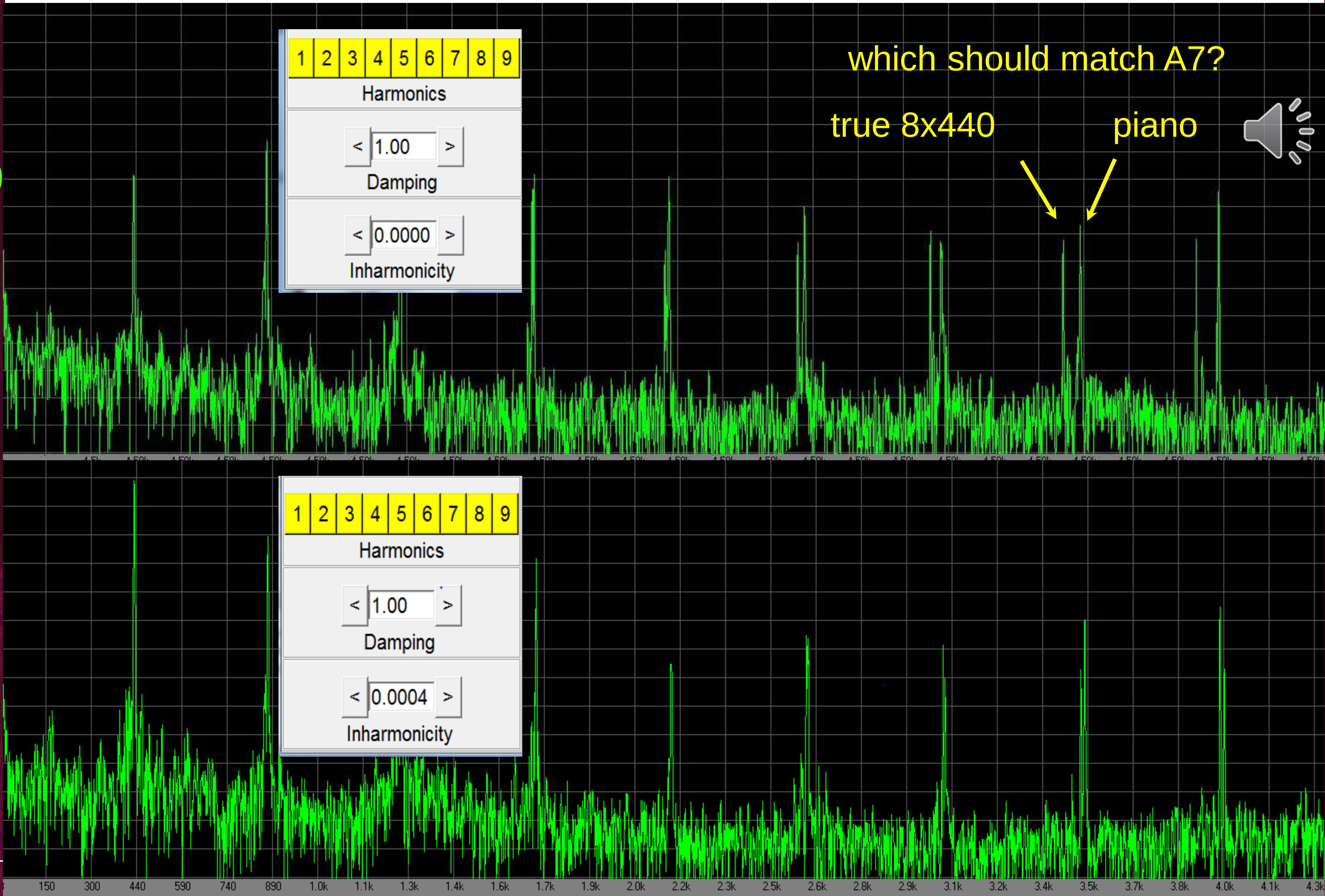
True Equal Temperament Frequencies

	0	1	2	3	4	5	6	7	8
C		32.70	65.41	130.81	261.63	523.25	1046.50	2093.00	4186.01
C#		34.65	69.30	138.59	277.18	554.37	1108.73	2217.46	
D		36.71	73.42	146.83	293.66	587.33	1174.66	2349.32	
D#		38.89	77.78	155.56	311.13	622.25	1244.51	2489.02	
E		41.20	82.41	164.81	329.63	659.26	1318.51	2637.02	
F		43.65	87.31	174.61	349.23	698.46	1396.91	2793.83	
F#		46.25	92.50	185.00	369.99	739.99	1479.98	2959.96	
G		49.00	98.00	196.00	392.00	783.99	1567.98	3135.96	
G#		51.91	103.83	207.65	415.30	830.61	1661.22	3322.44	
A	27.50	55.00	110.00	220.00	440.00	880.00	1760.00	3520.00	
A#	29.14	58.27	116.54	233.08	466.16	932.33	1864.66	3729.31	
B	30.87	61.74	123.47	246.94	493.88	987.77	1975.53	3951.07	

But first – a critical note about 'real' strings (where 'art' can't be avoided)

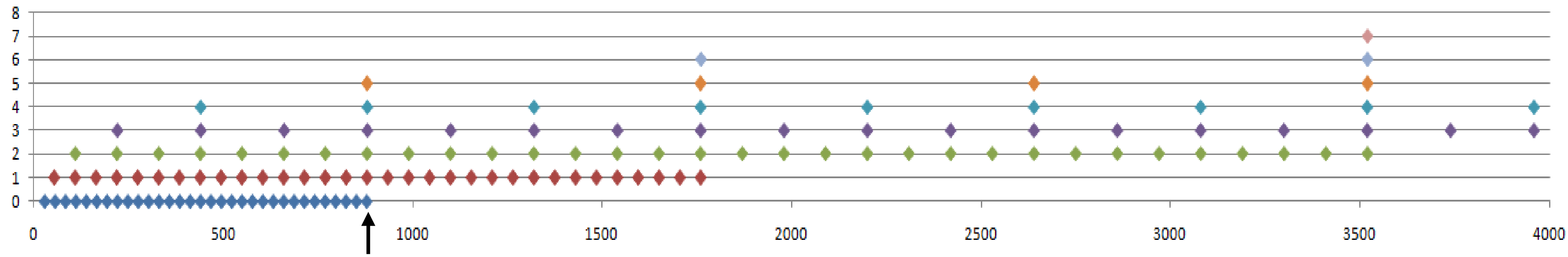
- strings have 'stiffness'
- bass strings are wound to reduce this, but not all the way to their ends
- treble strings are very short and 'stiff'
- thus harmonics are not true multiples of fundamentals
  – f_n is increased by a factor of $\sqrt{1+\beta n^2}$
- concert grands have less inharmonicity because they have longer strings

A4 (440) inharmonicity



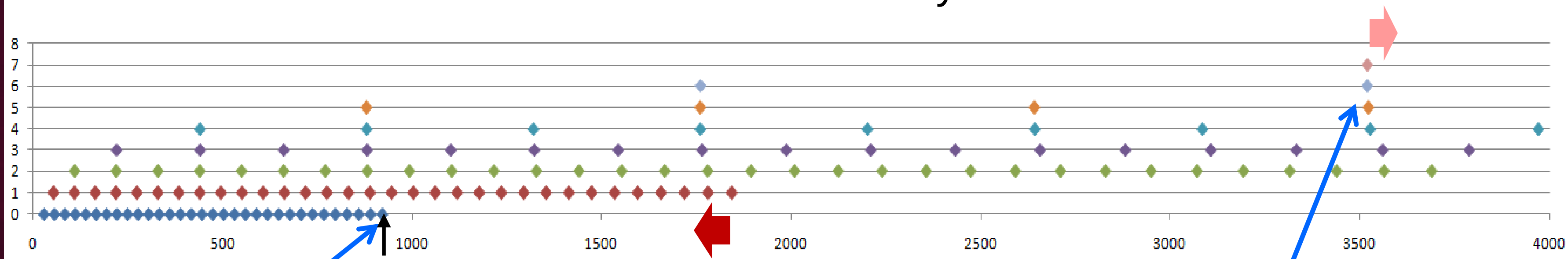
Piano Tuning

Tuning the 'A' keys: Ideal strings $f_0 = 440(2^n); n = -4 \cdots 2$



32 f_0

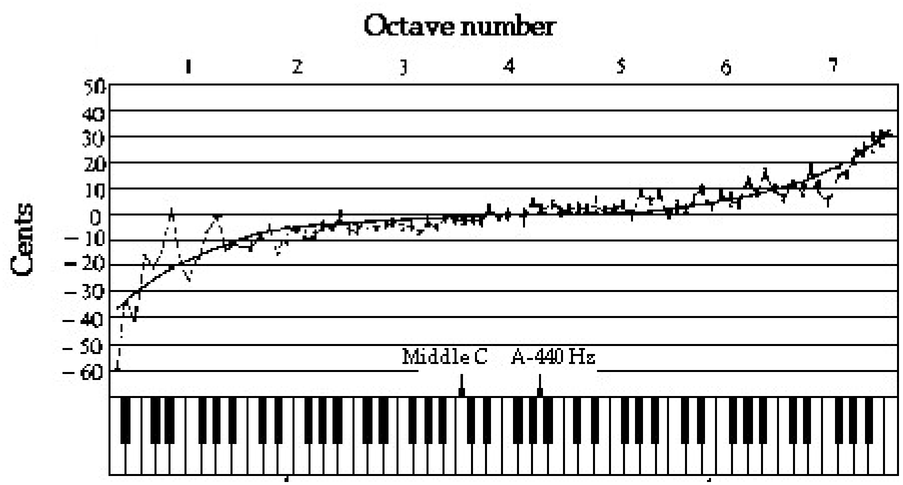
With 0.0001 inharmonicity



33.6 f_0

sounds 'sharp'

sounds 'flat'



Need to "Stretch" the tuning.

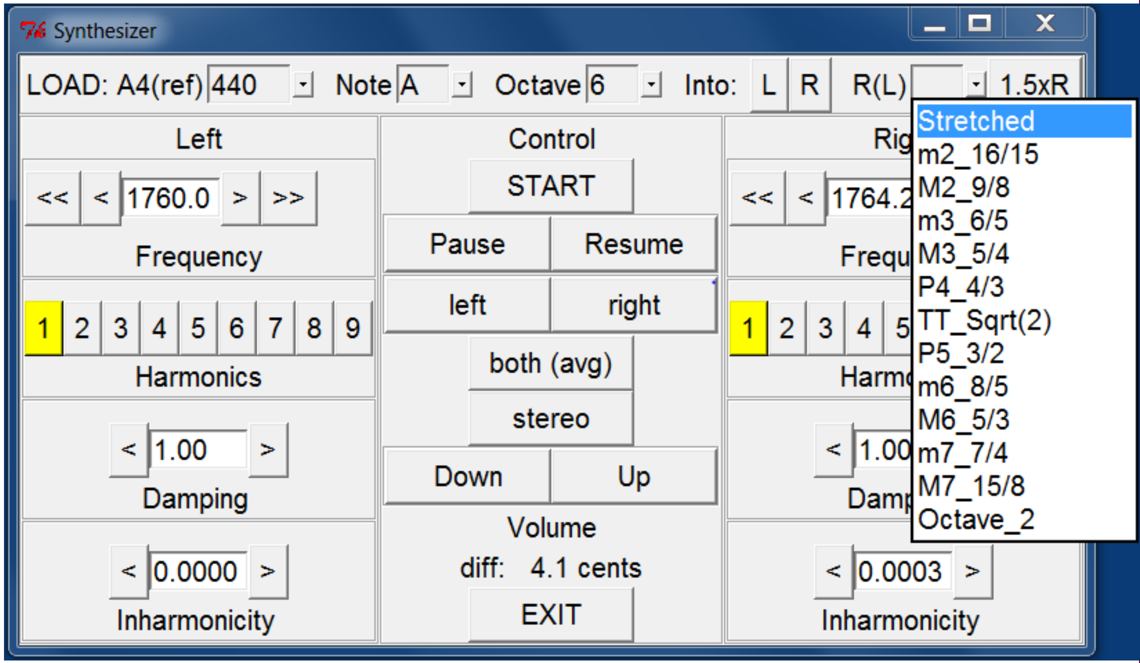
Can not match all harmonics, must compromise → 'art'

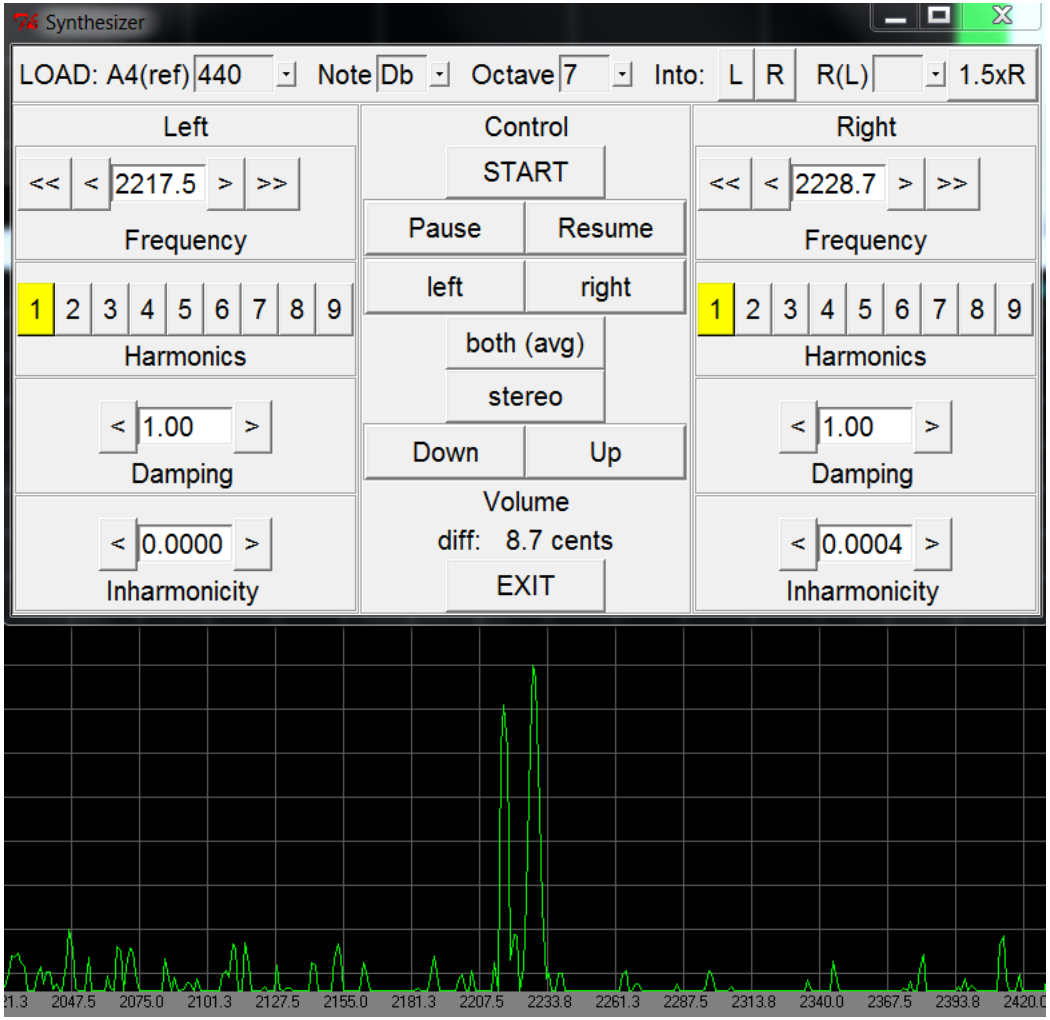
(how I've done it)

octaves 3-5: no stretch (laziness on my part)

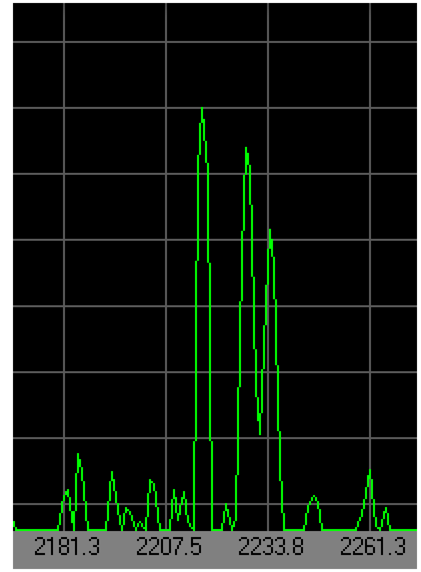
octaves 0-2: tune harmonics to notes in octave 3

octaves 6-7: set 'R' inharmonicity to ~0.0003
load note into L and use R(L) 'Stretched'

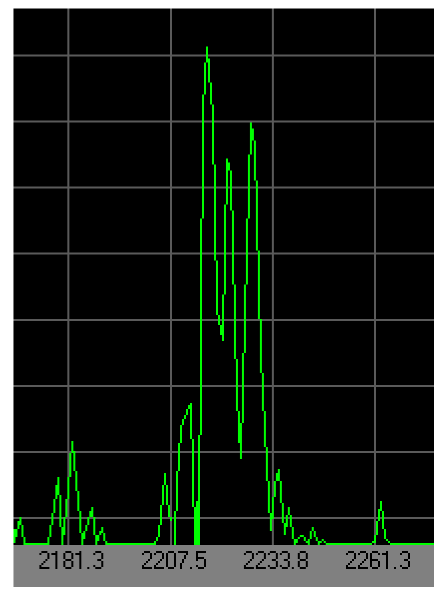




Trying to set D*b*7



With D*b*4



With D*b*5

The effect is larger for higher harmonics, and so you simply can't match everything at the same time.

but some keys don’t work…

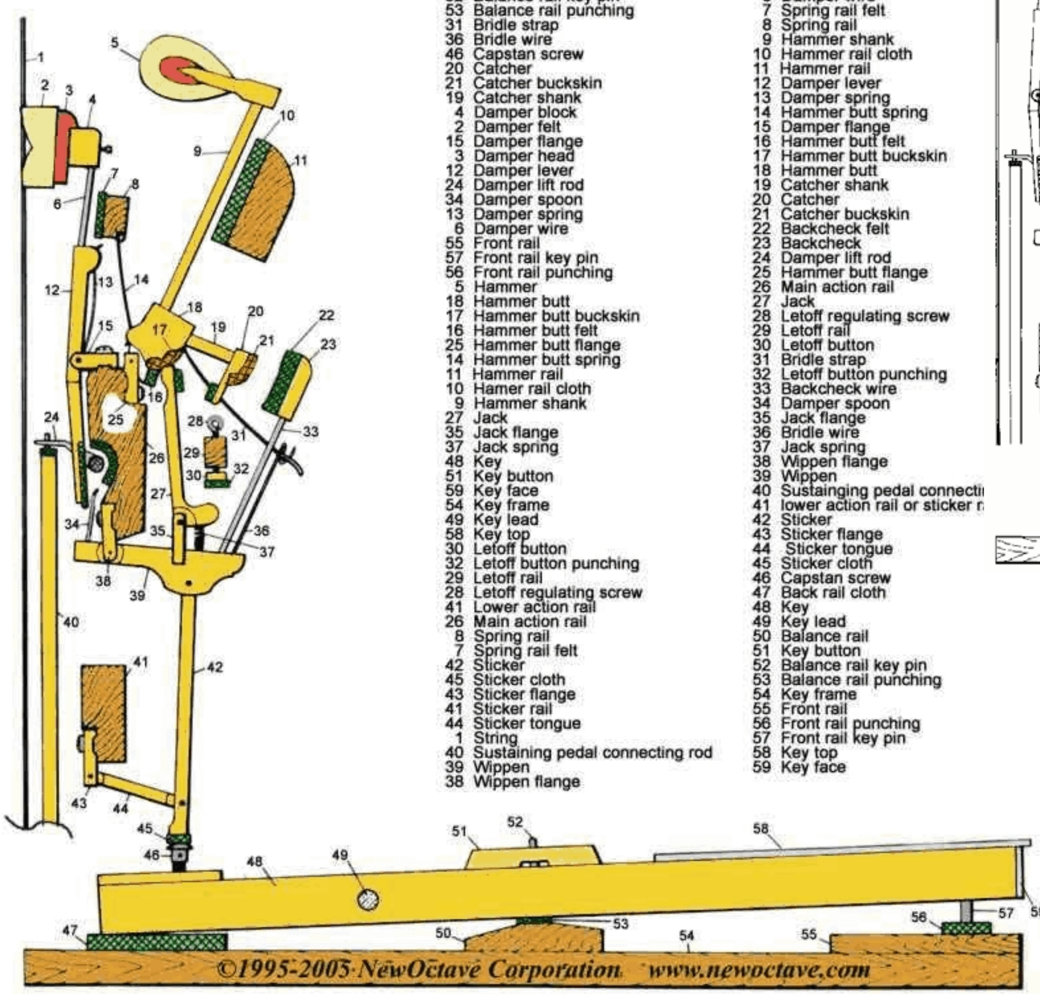
pianos were ***designed*** to come apart

(if you break a string tuning it,
you’ll need to remove the ‘action’ anyway)

(remember to number the keys before removing them
and mark which keys hit which strings)

“Regulation”
Fixing keys, and making mechanical adjustments
so they work optimally, and ‘feel’ uniform.

A clickable version of this image is now available at **pianoparts.com/upright**

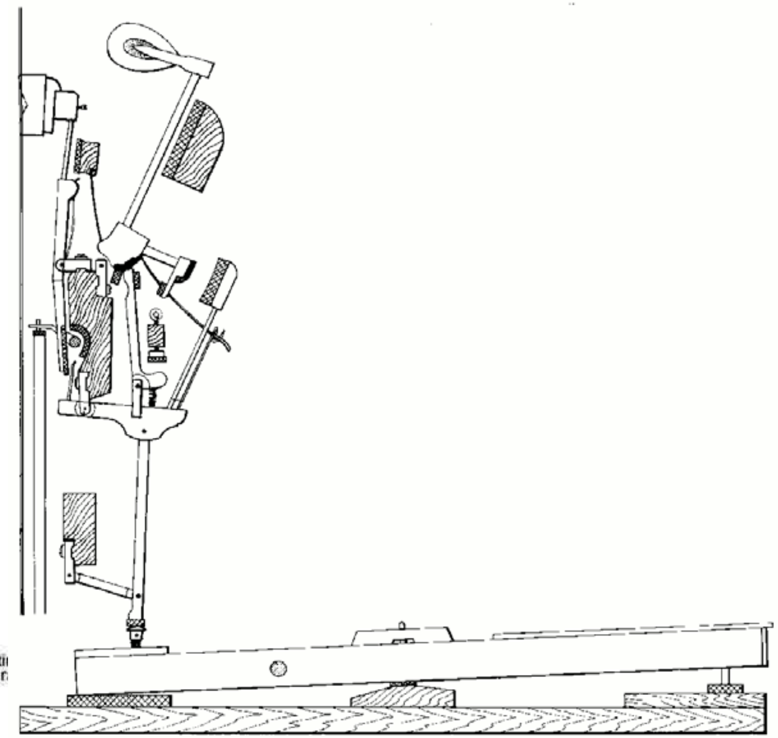


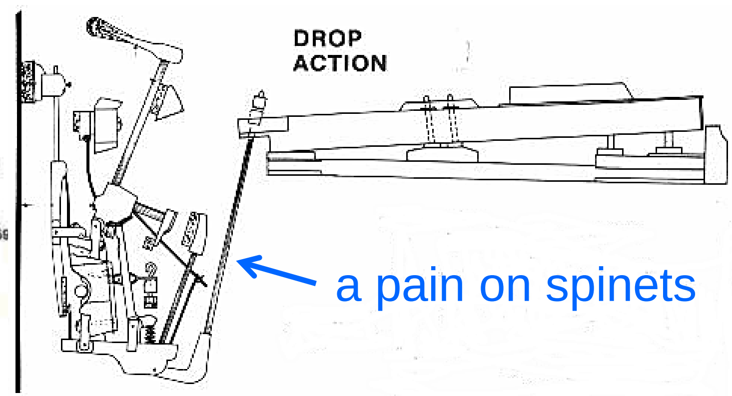
Alphabetically:

47 Back rail cloth
23 Backcheck
22 Backcheck felt
33 Backcheck wire
50 Balance Rail
52 Balance rail key pin
53 Balance rail punching
31 Bridle strap
36 Bridle wire
46 Capstan screw
20 Catcher
21 Catcher buckskin
19 Catcher shank
4 Damper block
2 Damper felt
15 Damper flange
3 Damper head
12 Damper lever
24 Damper lift rod
34 Damper spoon
13 Damper spring
6 Damper wire
55 Front rail
57 Front rail key pin
56 Front rail punching
5 Hammer
18 Hammer butt
17 Hammer butt buckskin
16 Hammer butt felt
25 Hammer butt flange
14 Hammer butt spring
11 Hammer rail
10 Hamer rail cloth
9 Hammer shank
27 Jack
35 Jack flange
37 Jack spring
48 Key
51 Key button
59 Key face
54 Key frame
49 Key lead
58 Key top
30 Letoff button
32 Letoff button punching
29 Letoff rail
28 Letoff regulating screw
41 Lower action rail
26 Main action rail
8 Spring rail
7 Spring rail felt
42 Sticker
45 Sticker cloth
43 Sticker flange
41 Sticker rail
44 Sticker tongue
1 String
40 Sustaining pedal connecting rod
39 Wippen
38 Wippen flange

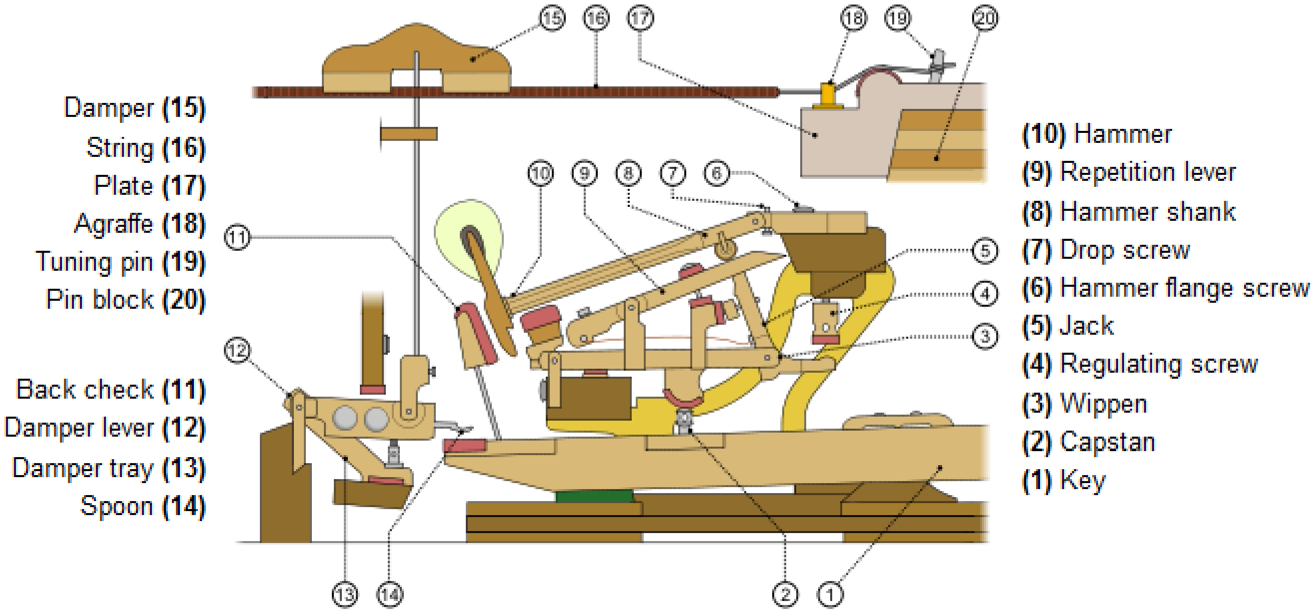
Numerically:

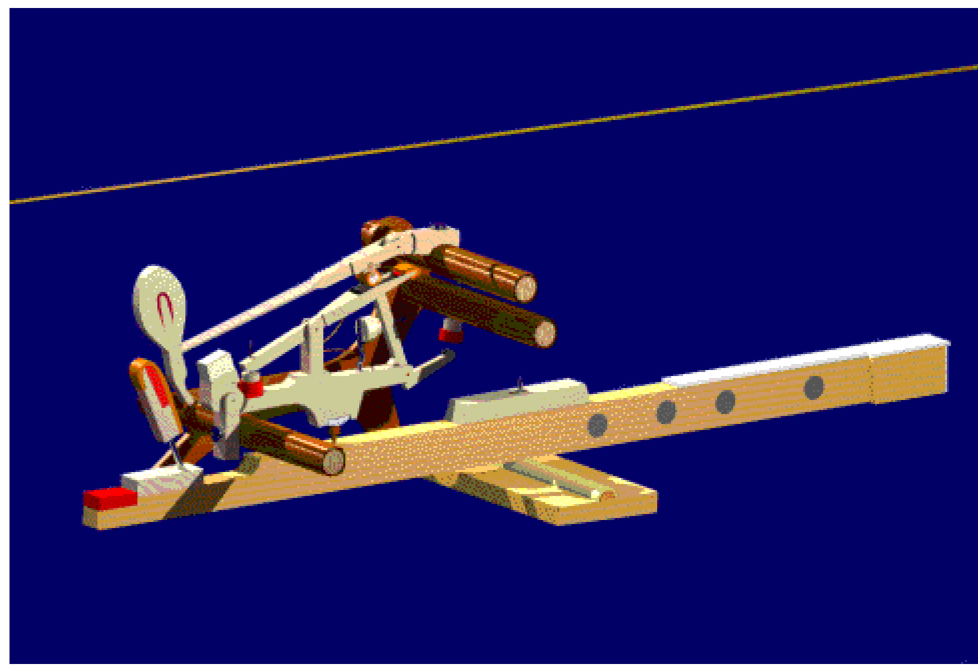
1 String
2 Damper felt
3 Damper head
4 Damper block
5 Hammer
6 Damper wire
7 Spring rail felt
8 Spring rail
9 Hammer shank
10 Hammer rail cloth
11 Hammer rail
12 Damper lever
13 Damper spring
14 Hammer butt spring
15 Damper flange
16 Hammer butt felt
17 Hammer butt buckskin
18 Hammer butt
19 Catcher shank
20 Catcher
21 Catcher buckskin
22 Backcheck felt
23 Backcheck
24 Damper lift rod
25 Hammer butt flange
26 Main action rail
27 Jack
28 Letoff regulating screw
29 Letoff rail
30 Letoff button
31 Bridle strap
32 Letoff button punching
33 Backcheck wire
34 Damper spoon
35 Jack flange
36 Bridle wire
37 Jack spring
38 Wippen flange
39 Wippen
40 Sustainging pedal connecti
41 lower action rail or sticker r
42 Sticker
43 Sticker flange
44 Sticker tongue
45 Sticker cloth
46 Capstan screw
47 Back rail cloth
48 Key
49 Key lead
50 Balance rail
51 Key button
52 Balance rail key pin
53 Balance rail punching
54 Key frame
55 Front rail
56 Front rail punching
57 Front rail key pin
58 Key top
59 Key face





Action of a grand piano





“Voicing” the hammers

NOT for the novice
(you can easily ruin a set of hammers)

Let’s now do it for real…

pin turning
unisons (‘true’ or not?)
tune using FFT
put it back together