PHYS5455 Quantum Mechanics: Fall 2004
Homework 4, due Saturday October 2, 10:00 am

Question 1: Classical Poisson brackets and some symmetry operations.

Let us consider a system in 3D described by rectangular coordinates: $\vec{x}_i,\ (i = 1, 2, ..N)$ and corresponding conjugate momentum $\vec{p}_i,\ (i = 1, 2, ...N)$. Then consider a physical quantity A which is a function of $\vec{x}_i$ and $\vec{p}_i$: $A = A(\vec{x}_1, \vec{x}_2, ..; , \vec{p}_1, \vec{p}_2, ...)$.

(i) If the system is displaced uniformely by an infinitesimal amount $\delta\vec{x} = \vec{\epsilon}$, demonstrate that the change in A up to the first order in $\vec{\epsilon}$ is equal to

$$\delta A = \vec{\epsilon} \cdot \{A, \vec{p}\},$$

with $\vec{p} \equiv \sum_{i=1}^{N} \vec{p}_i$ the total momentum of the system.

(ii) If the system is rotated by an infinitesimal angle $\delta\phi$ around an axis denoted by a unit vector $\vec{n}$, hence by $\delta\vec{\phi} \equiv \vec{n}\delta\phi$, demonstrate that the corresponding change in A is

$$\delta A = \{\delta\vec{\phi} \cdot \vec{L}, \sum_{i=1}^{N} \vec{x}_i\} \cdot \vec{\nabla}_i A + \{\delta\vec{\phi} \cdot \vec{L}, \sum_{i=1}^{N} \vec{p}_i\} \cdot \vec{\nabla}'_i A,$$

where the first gradient operator is with respect to $\vec{x}_i$ while the second one is with respect to $\vec{p}_i$. In the above expression $\vec{L} = \sum_{i=1}^{N} \vec{L}_i$ is the total angular momentum of the system.

[Hint] Use the result of Q1 in the last homework, and the fact that under the given infinitesimal rotation, and the change in $\vec{x}_i$ and $\vec{p}_i$ are $\delta\vec{\phi} \times \vec{x}_i$ and $\delta\vec{\phi} \times \vec{p}_i$, respectively (to be discussed in Tuesday, Sept. 28th's lecture, or look at the handout on Tuesday, Sept. 21st).

Question 2: Polar representation of wave functions, and classical limit.

Recall that an ordinary complex number may be expresses in a polar form: $z = r\exp{(i\theta)}$, where r and θ are real variables. We adopt the same representation for solutions to the Schrödinger equation: consider a particle in 3D with a mass m in a real-valued potential $V(\vec{x})$. Then let the solution to the corresponding time-dependent Schrödinger equation be $\psi(\vec{x}, t)$ and express it in a polar form:

$$\psi(\vec{x}, t) \equiv R(\vec{x}, t)\exp{[iS(\vec{x}, t)/\hbar]},$$

where both R and S are real functions depending both on $\vec{x}$ and t. Note S has the dimension of action.

(i) For the simplest case: a free particle with momentum $\vec{p}$ and corresponding energy E, identify R and S. (NOTE: this does not apply to the subsequent questions below where $V(\vec{x})$ is NOT zero).

(ii) Find the probability density $\rho(\vec{x}, t)$ and the probability current density $\vec{j}(\vec{x}, t)$ in terms of $R(\vec{x}, t)$ and $S(\vec{x}, t)$. Then write the corresponding continuity equation in terms of these functions.

(iii) Express the time dependent Schrödinger equation explicitly in terms of these real functions. Find a common factor which should be $\exp{(iS/\hbar)}$ on both sides of the equation, and cancel it to make the expression simpler.

(iv) Separate the result obtained in (iii) into real and imaginary parts, thereby obtain two differential equations. Then explicitly demonstrate that the equation from the thus separated imaginary part appears to have no explicit dependence on $\hbar$, whereas the one from the real part does have an explicit $\hbar$ dependence.

(v) Estabilsh the relation between the result from (ii) above and the imaginary part of the equation as mentioned in (iv) above.

(vi) Pick up the real part of the equation from the one discussed in (iv). Then find the equation resulting from the limit: $\hbar \to 0$. By identifying the classical momentum to be $\vec{p} \equiv \vec{\nabla}S$, how does your result compare with the classical Hamilton-Jacobi equation discussed in the lecture? Finally, show that the result obtained in (v) has a clearer physical picture by this identification of momentum.

[Note]: By recalling the quantum criteria discussed in the class earlier in the semester, the $\hbar \to 0$ limit discussed above is equivalent to the situation where physics is no longer quantum but classical.