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The renormalisation group approach to scaling in physics

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Abstract

We review the application of the renormalisation group to many-body problems, with particular emphasis on the scaling behaviour which may arise. An introductory section discusses the difficulties of calculating the many-body effects near a magnetic phase transition, and shows by a simple example how the renormalisation group may be used to predict scaling behaviour. A general discussion of scaling and universality follows, with a review of techniques for calculation in lattice models, and an outline of the approaches in field theory. The range of applications of these techniques is discussed. The introductory sections are designed for the non-expert.

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Contents

1. Introduction

1.1. Scaling in many-body problems

Scaling, or scale invariance, is not a typical phenomenon of physics. For most aspects of all physical systems, adequate theoretical understanding may be attained by concentrating on only a small range of length (and/or time, mass, etc) scales spread about some characteristic point. For example, to comprehend Galileo's experiment at the Tower of Pisa, it was not necessary to have an understanding of the motion of the molecules which make up the balls. Similarly, excited states of the nuclei need not be considered in order to study adequately the structure of a molecule, and planetary motions are ignored in the analysis of the evolution of galaxies, and so on. These examples share one common feature: the picture, to be described theoretically by, say, an effective Hamiltonian, changes as the investigator changes his attention from one characteristic length to another; these phenomena are scale-dependent.

However, it appears now that there are many phenomena exhibiting scale invariance and for which the best physical description is in terms of a picture (Hamiltonian) which remains the same under a change of length scale. Major examples are (i) in statistical physics, the critical point of a system undergoing a (second-order) phase transition and analogous phenomena and (ii) in elementary particle physics, the limit of collision energy very much larger than typical hadron masses. In this review we shall concentrate on the former, where enormous progress has been made since the previous review in *Reports on Progress in Physics* (Fisher 1967a).

In scale-invariant systems, one common feature poses as the major obstacle for ordinary analysis: the essential participation of degrees of freedom associated with all length scales. For example, in a statistical system at criticality, correlated fluctuations of all sizes are equally important. Ordinary analysis fails in these situations since it singles out only a few 'important' degrees of freedom. Such methods are perfectly adequate in, say, the first example given: two degrees of freedom (vertical position and velocity of the centre of mass) enter essentially while the 10^{20} or so degrees of freedom associated with molecular motion can be safely ignored.

Until recently, progress on a description of criticality has come from mathematical or computational dexterity—exact results for specific models and high/low temperature series techniques. The renormalisation group (RG) analysis, by contrast, attempts to provide an understanding of the physics by actually taking into account these many-body effects, if only approximately. The central idea is a step-by-step method. In each step, typically, a fraction of the many degrees of freedom is eliminated while the effects of such a decimation are translated into an effective Hamiltonian for the remaining degrees of freedom. Unlike ordinary analysis, which generally *ignores* the effects of the degrees of freedom associated with all scales except a chosen few, this method *incorporates* some of the effects into the 'new' Hamiltonian at each step. This action of generating a new Hamiltonian, $\mathscr{H}'(s')$, for the new degrees of freedom (s') from the old $\mathscr{H}(s)$ is called a renormalisation group transformation.

The sequence of effective Hamiltonians obtained after a large number of such steps may in principle exhibit an enormous range of behaviours. Perhaps the simplest

possibility, and the centre of attention in this review, is convergence towards a fixed point $\mathscr{H}^*$. When $\mathscr{H}^*$ is approached there is no longer a change in the picture ($\mathscr{H}^*$) under a RG transformation. Since a typical RG transformation involves a change of scale, in that the density of degrees of freedom is changed, scale invariance and the existence of $\mathscr{H}^*$ are intimately related.

The RG was introduced first in the context of quantum electrodynamics by Stueckelberg and Petermann (1953) and by Gell-Mann and Low (1954) to study the high-energy behaviour of interacting electrons and photons. This formal development lay dormant for many years. The above picture of thinning of degrees of freedom was applied by Kadanoff (1966) to predict qualitatively scaling behaviour at second-order phase transitions. The relevance for phase transitions of the earlier quantum field theory methods was realised (Di Castro and Jona-Lasinio 1969) and was implemented quantitatively by Larkin and Khmel'nitskii (1969) for phase transitions in uniaxial ferroelectrics. (These have the special property that the attractive fixed point corresponds to zero interactions.) The application of the RG to the Kondo problem was instigated by Anderson *et al* (1970). In a series of papers, Wilson (1971a, b) and Wilson and Fisher (1972) showed how Kadanoff's ideas could be implemented using field theory techniques and provided a quantitative tool for calculations of critical behaviour in many systems. In particular the discovery of the ϵ expansion (Wilson and Fisher 1972) opened the way for the use of the highly developed techniques of renormalised perturbation theory (Brézin *et al* 1973a, b, c, d, Mitter 1973, Schroer 1973). A quantitative realisation of Kadanoff's ideas in Ising-like lattice models was given by Niemeijer and van Leeuwen (1973, 1974). The spin-$\frac{1}{2}$ Kondo problem was solved (Wilson 1974b, 1975). References to results in elementary particle physics (which are on the whole less well established experimentally) will be given later. Enormous numbers of applications of these developments have been made in the last five years.

1.2. Outline

An illustration of RG methods is presented in §2. A brief survey is given of the main features of critical phenomena. We show why ordinary analysis is inadequate and how RG methods can succeed. A simple example is worked out in some detail to provide a taste of the RG method.

In §3, we study the stability properties of an $\mathscr{H}^*$ and present the important links between these properties and physically measurable quantities: equation of state, correlations, etc. This section deals with the question: what can be calculated?

How to do calculations is dealt with in §§4 and 5. We review real-space RG methods in Ising-like lattice models in §4. To give an account of field theory methods from first principles is beyond the scope of this review. We give only a skeleton review of them in §5 and indicate a number of comprehensive references.

The sixth section is a brief survey of the immense field of applications. It is obviously impossible to give details—our aim is to review what has been done and to direct the reader to the main references on specific systems.

A newcomer to this subject should find most of the early sections (§§2–4) readable. Section 3.2 is more technical and to gain facility with the techniques in §5 requires investment in field theory methods, particularly the Feynman graph expansion. We hope that each subsection in §6 can be read independently of the other subsections.

2. The renormalisation group—a simple example

In this section we are going to illustrate by a very simple calculation the use of the renormalisation group (RG) in a magnetic phase transition. We shall discuss the physics of the problem, write down a theoretical model, describe a simple way of implementing the RG idea in the model and show how a description of the physics is obtained in the RG formalism. We stress that none of these topics is covered in an exhaustive fashion in this section; our aim is to give as simple an illustration as possible of the RG in action. References will be given in the summary at the end of the section.

2.1. Magnetic phase transition

The magnetism that we see in materials such as iron and nickel is understood to result from the cooperative alignment of the microscopic magnetic moments, or spins, of electrons in the material. The forces which make the spins align have their origin in the exchange interaction in quantum mechanics. The strength of these aligning forces determines the temperature up to which the material has permanent magnetisation. A typical graph of magnetisation M as a function of temperature T is shown in figure 1. In the absence of an applied magnetic field the magnetisation decreases smoothly and vanishes at the Curie or critical temperature T_c. This disappearance of magnetisation is understood to arise not from the disappearance of the microscopic magnetic moments of the electrons but from the reduction of the amount of alignment of their spins. If we lower the temperature through the Curie temperature, the alignment reappears. (Strictly speaking, the alignment occurs spontaneously only within domains, formed to minimise the dipole interactions between the spins. An external magnetic field H must be applied to align the domains and 'magnetise' the material. The curves in figure 1 should be regarded as behaviour in large single domains.)

We say that the system goes through a phase transition at the critical point ($H=0$, $T=T_c$) changing from being paramagnetic ($T>T_c$) to ferromagnetic ($T<T_c$). Technically this is classified as a second-order phase transition, characterised by the absence of latent heat and the continuous growth from zero of an order parameter, the spontaneous magnetisation in this case. Not all magnetic phase

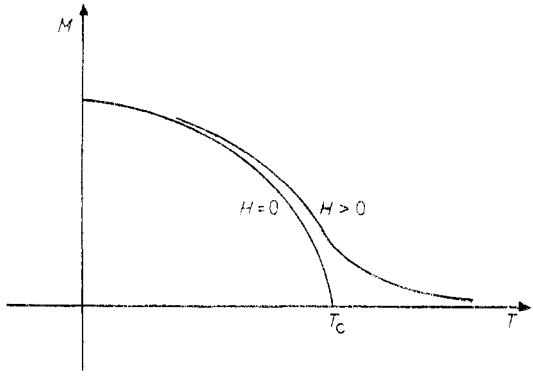

Figure 1. Dependence of magnetisation on temperature in an ideal ferromagnet.

transitions are of this type but many other examples of this kind of transition will be discussed in §6.

The theoretical study of the critical behaviour, where H is small and T is close to T_c, has aroused enormous interest for many years, both because of the nature of the observed macroscopic phenomena and because of the microscopic picture which has been developed to understand the onset of the phase transition as the temperature is lowered to T_c.

In order to appreciate this latter point, let us consider a paramagnetic material at very high temperatures. In this region, the spins of the electrons act essentially independently of one another, just as do the molecules in a gas at sufficiently high temperature and low density. As the temperature is lowered, the spins begin to respond more to the forces aligning them with their neighbours in the solid. Therefore a correlation develops in which neighbouring spins tend on the average to be parallel to one another. As the temperature is lowered further, the range of this correlation, called the correlation length ξ, becomes larger. In the absence of an applied field H there is not yet any magnetisation of course, because these aligning effects still average out to zero over the solid; only over distances less than about ξ does the material appear inhomogeneous, because of these fluctuations. The crucial concept is that as T is lowered to T_c (with $H=0$) in an idealised system, the correlation length ξ tends to *infinity*, marking the onset of the 'infinite'-range correlation of spontaneous magnetisation. The principal idealisation is that the material should be infinitely large, with all domain effects negligible; 'infinitely large' is realised rather well in practice in units of atomic spacing.

Those more familiar with the liquid–gas system will appreciate that the phenomenon of critical opalescence signals dramatically the appearance of large correlated fluctuations close to the critical point of the system.

The above remarks summarise very briefly the difficulty of obtaining a theoretical description of such critical behaviour; each spin interacts directly only with the spins in its neighbourhood and the phase transition occurs through the cooperative effect of the interactions among arbitrarily large numbers of the spins. This is typical of the kind of many-body problem which the renormalisation group is specifically designed to tackle.

There is now an established (idealised, of course) format which is used to display the results of experiments on second-order phase transitions. A more detailed discussion of the physical phenomena is given in §3. Here we mention only two primary features.

(*a*) *Critical exponents.* The curve of spontaneous magnetisation in figure 1 can be described close to T_c by a relationship of the form

$$M \propto (T_c - T)^{\beta} \qquad (H=0,\ T<T_c). \tag{2.1}$$

β is an example of a critical exponent. A typical value is about 0·37. Many other exponents can be defined describing the behaviour of other quantities close to the critical point. For example, the exponent ν describes how the correlation length ξ goes to infinity as T is lowered to T_c (with $H=0$):

$$\xi \propto (T - T_c)^{-\nu} \qquad (T>T_c,\ H=0). \tag{2.2}$$

We shall illustrate later in this section how this behaviour is predicted and ν is calculated using the renormalisation group.

The relations (2.1) and (2.2) exhibit scaling behaviour, in the sense that e.g.

equation (2.1) is invariant under the scale changes $(T-T_c)\to\lambda(T-T_c)$, $M\to\lambda^\beta M$. This *scaling property* is displayed more dramatically in the equation of state, for example, which involves a function of two variables in general, e.g. $H=H(M, T)$. It is found that in the critical region the equation of state can be expressed as a function of one variable, in the form

$$H/M^\delta = h[(T-T_c)/M^{1/\beta}].$$

This behaviour will be discussed in more detail in §3.

(*b*) *Universality.* It is clear that some of the parameters describing the phase transition do depend on the detailed nature of the forces between the electron spins; the critical temperature is higher the stronger the aligning forces. However, it is remarkable that many of the properties near a phase transition point are *universal.* Critical exponents such as β and ν (as well as the function h up to choice of units) appear to depend only on qualitative features of the system which are easily characterisable, such as d, the number of spatial dimensions, the underlying symmetry and the range of interactions. For example, all uniaxial three-dimensional magnets with predominantly short-range interactions have the same values for critical exponents. There is even evidence that the critical behaviour in liquid–gas systems is also universal in this sense, with the *same* critical exponents as in these uniaxial magnets (for the appropriate thermodynamic analogues).

We shall see the origin of such universality in the renormalisation group in the next section.

We postpone any further discussion of these two remarkable phenomena, stressing only that their appearance is to be associated with the existence of the macroscopic fluctuations which occur near the critical point.

2.2. *Theoretical framework*

2.2.1. The Ising model. The standard theoretical framework for studying the thermodynamic properties of a system is statistical mechanics. In this framework, the average values of thermodynamic quantities, which are what we observe in experiments, can be calculated in principle from the forces between the microscopic constituents of the system (e.g. the electron spins in the magnet).

There are always three basic ingredients in statistical mechanics. We shall discuss these for the simplest of all models, the Ising model. This model is used as a prototype model for describing the critical behaviour of uniaxial magnets with short-range interactions, as discussed at the close of the previous subsection. It is not claimed that the model is a realistic one for any particular such uniaxial magnet, but that it contains the correct *qualitative* features of the interactions between spins in *all* such magnets so that it is expected also to have the same critical exponents as all such magnets, according to the universality principle.

(i) The first ingredient is a choice of microscopic variable, to correspond to an electron spin in the magnet. The Ising model has a prototype spin variable s which takes only two values (± 1) at each site, labelled by m, of a rigid lattice as in figure 2. Thus an Ising model for a system of N lattice sites has N spin variables s_m.

(ii) The second ingredient is the Hamiltonian: this is the object which gives the energy of the system for a given set of values, or configuration, of the spins. The

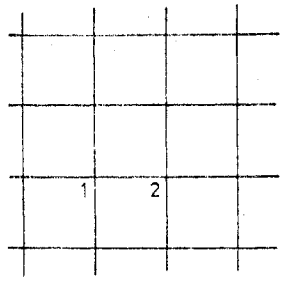

Figure 2. Part of a rigid lattice on which the Ising model can be defined; 1 and 2 are nearest-neighbour sites.

typical Hamiltonian we shall consider has the form

$$\mathscr{H} = -J \sum_{\langle mn \rangle} s_m s_n \tag{2.3}$$

where $\Sigma_{\langle mn \rangle}$ denotes the sum over all pairs of sites m and n which are nearest neighbours to one another on the lattice, e.g. sites 1 and 2 in figure 2. We say that this is a nearest-neighbour interaction, and J is the nearest-neighbour coupling constant, measuring the strength of this interaction.

Clearly the contribution to the sum in equation (2.3) from two parallel (anti-parallel) adjacent spins is $-J(+J)$. Therefore, provided J is positive, the lowest energy configuration is one in which all the spins are parallel (say, all are $+$) and therefore at low temperatures the system should look ferromagnetic. On the other hand, if J is negative, the configuration of lowest energy occurs when all adjacent spins are antiparallel, corresponding to the antiferromagnetic state of figure 3. To summarise, if a phase transition takes place we expect:

$J > 0$ ferromagnetic

$J < 0$ antiferromagnetic.

The conventional interaction which is added to the Hamiltonian (2.3) to describe the effect of a uniform external magnetic field H is

$$\mathscr{H}_H = -H \sum_m s_m.$$

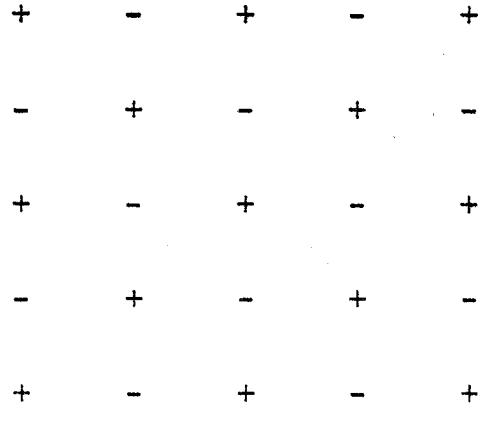

Figure 3. $J < 0$: antiferromagnetic ground state.

(iii) The third ingredient is a prescription for calculating the expectation values of thermodynamic quantities. These are given in terms of the correlation functions, defined by

$$\langle s_{m_1} s_{m_2} \ldots s_{m_p} \rangle = \left[\sum_{\{s\}} s_{m_1} s_{m_2} \ldots s_{m_p} \exp(-\mathscr{H}/kT) \right] \left[\sum_{\{s\}} \exp(-\mathscr{H}/kT) \right]^{-1}. \tag{2.4}$$

Here $\exp(-\mathscr{H}/kT)$ is the conventional Boltzmann factor, which weights the sum over all configurations of the spins, $\Sigma_{\{s\}}$. If there are N lattice sites, there are 2^N terms in this sum, since each spin can take on two values.

The denominator in equation (2.4) is the partition function:

$$Z = \sum_{\{s\}} \exp(-\mathscr{H}/kT). \tag{2.5}$$

The mean magnetisation per lattice site, which is what is measured experimentally, is identified as

$$M = \frac{1}{N} \sum_{m} \langle s_m \rangle \tag{2.6}$$

or simply as $\langle s_m \rangle$ for any site m if translation invariance holds. The higher correlation functions are related to physical quantities; we shall discuss this further in the next section.

Strictly speaking, one should always consider a system of a finite number N of lattice sites and apply some appropriate boundary conditions to the spins on the surface. One then obtains the behaviour of a macroscopic system by taking the 'thermodynamic limit' $N \to \infty$. In fact, it is only in this limit that any singularities such as equation (2.1) can occur; for a system of a *finite* number N of lattice sites, the expression (2.6) is a finite sum of finite functions which are entire (analytic) for all positive T. This remark is clearly related to the idea that the phase transition occurs when the correlation length is infinite which can occur only in an infinite system. In this review the problems of taking the limit $N \to \infty$ rigorously will be side-stepped as we shall see.

2.2.2. Conventional calculations in the Ising model. (*a*) Exact solutions. In principle, equations (2.3), (2.4) and (2.6) enable one to calculate exactly the magnetisation in this model. In practice, the configuration sums in (2.4) cannot be performed exactly in the limit $N \to \infty$, except in a very small number of exceptional cases. These include (i) one-dimensional systems, where one finds that no transition takes place at any finite temperature (for interactions $\Sigma_{m,n} J_{mn} s_m s_n$ where J_{mn} decreases faster than $|n-m|^{-\alpha}$, $\alpha > 2$); and (ii) Onsager's solution, and generalisations of it, for the Hamiltonian (2.3) in two dimensions. The result does exhibit a phase transition, with scaling behaviour (2.1) and the critical exponent $\beta = \frac{1}{8}$. This kind of work till now has been restricted to two-dimensional models, with only the simplest kinds of interactions; there is little insight into the phenomenon of universality, and no solutions are known when an external field is added.

(*b*) Approximation schemes. In the absence of an exact solution, the theorist naturally looks for a suitable approximation scheme to provide a description of the region of interest. There are many such approximation schemes for the equations

(2.3)–(2.6). We mention the high-temperature series expansion which consists, in its most primitive form, of expanding the Boltzmann factor for T very large:

$$\exp(-\mathscr{H}/kT) = 1 - \mathscr{H}/kT + \frac{1}{2!}(\mathscr{H}/kT)^2 + \ldots . \tag{2.7}$$

The configuration sum can now be performed on each of the terms in this expansion, and the correlation functions are obtained as a systematic expansion in powers of $1/T$.

However, it is typical of such approximation schemes that they break down in the critical region. Their failure in this region can be traced back to the existence of the large correlation length. For example, at first order in $1/T$, only one power of the Hamiltonian is present and the correlation between spins extends only over the range of the interactions in $\mathscr{H}$. At second order in $1/T$, two powers of $\mathscr{H}$ are present and the correlation between spins extends over twice the range of the interactions in $\mathscr{H}$, and so forth. In order to obtain infinite-range correlation, as at the critical point, one must work, in principle, to infinite order in the high-temperature series expansion. In fact, the radius of convergence of this expansion is certainly bounded by $1/T_c$. For example, the spontaneous magnetisation is zero to all orders in the expansion in $1/T$; this expansion must clearly break down for $1/T > 1/T_c$ ($T < T_c$). Other expansion schemes suffer from the same defect. (The breakdown may be even more dramatic; the first-order 'correction' to the mean field approximation is itself infinitely larger than the mean field behaviour in the critical region. This is discussed further in §5.4.) However, it should be said that computer studies of series expansions, and Monte Carlo methods enable one to extract good numerical results for critical behaviour, which support the features of universality and critical exponents and provide a convenient yardstick against which to measure the results of RG methods.

2.3. The renormalisation group

2.3.1. Basic idea. In the magnetic phase transition, we are clearly faced with a difficult problem. Each spin interacts directly only with the spins in its neighbourhood and the phase transition occurs through the cooperative effects of the interactions among a large number of spins. The existence of these cooperative effects precludes the direct use of approximation schemes which take into account only interactions amongst a few spins.

The renormalisation group is designed specifically to tackle this kind of problem. There are many ways of formulating the renormalisation group, but all have the same basic ingredient. In the language of spins on a lattice, such as in the Ising model, the central idea is a stepwise reduction in the number of spins in a region the size of the correlation length. At each step, one eliminates several neighbouring spins in favour of a single new spin which is made to reproduce the effect of the old cluster. After an appropriate number of steps, we are left with a system in which the ratio of correlation length to spin separation is not large. Such a system is far away from its critical point and we can do any suitable kind of perturbation theory to finish off the problem.

In order to ensure that a single new spin does indeed play the role of several of the previous ones, the Hamiltonian of the new spins must be specially chosen; the interactions it contains must 'mock up' the interactions of all the previous spins.

The parameters, or coupling constants, of the new Hamiltonian will therefore depend on the coupling constants which parametrise the interactions of the initial system.

In order to simplify the notation, we absorb the factor $1/kT$ into the Hamiltonian $\mathscr{H}$ and denote the various kinds of interactions by K_i (or $\boldsymbol{K}$), e.g.

$$K_1 = J/kT \tag{2.8}$$

is the nearest-neighbour coupling, K_2 is the next-nearest-neighbour coupling, etc. In this notation, therefore, we write

$$K_i^{(1)} = f_i(\boldsymbol{K}) \qquad i = 1, 2, \ldots \tag{2.9}$$

where $K_i^{(1)}$ are the couplings of the new spins (the superscript (1) tells us that we have taken one reduction step).

At the second step one starts with a set of spins whose interactions are parametrised by $\boldsymbol{K}^{(1)}$ and repeats the process by introducing a second set of spins whose interactions are parametrised by $\boldsymbol{K}^{(2)}$ according to

$$K_i^{(2)} = f_i(\boldsymbol{K}^{(1)}).$$

One iterates the process, thinning the number of spins at each step and generating new equivalent Hamiltonians, parametrised at the lth step, say, by couplings $\{K_i^{(l)}\}$ where

$$K_i^{(l)} = f_i(\boldsymbol{K}^{(l-1)}). \tag{2.10}$$

As l increases, the systems described by $\{K_i^{(l)}\}$ are further and further from the critical point, in the sense that a region the size of the correlation length contains fewer and fewer independent spin variables. One stops the iterations at some suitable value of l, say L, for which there are only a few spins left in a correlation length and some conventional perturbation theory may be applied. This solves the original problem; it gives the properties of the original system in terms of the parameters $\{K_i^{(L)}\}$ and hence in terms of the original couplings $\{K_i\}$, as required, through the recursion formulae (2.10) which give the $\{K_i^{(L)}\}$ in terms of $\{K_i\}$.

One further point is worth stressing before we illustrate the RG transformation (2.9) in a simple example. In practice, the functions f_i can no more be calculated exactly than the original Hamiltonian can be exactly solved. One has to resort to some suitable approximation scheme. However, in contrast to thermodynamic quantities which are *singular* in the critical region, the functions f_i are expected to be *regular*. (It is possible to justify this expectation *a posteriori*, or by physical arguments.) The approximation schemes which failed for calculating thermodynamic quantities are therefore potentially usable here. This idea is a very important part of the RG method.

2.3.2. An example using decimation. Consider an Ising lattice in two dimensions, as in figure 2. The decimation scheme consists of dividing the lattice into a set of spins $\{\sigma\}$, which constitute a lattice of the same form as (isomorphic to) the original square, and the complementary set $\{\bar{\sigma}\}$. The example we shall consider is shown in figure 4. One then *defines* an effective Hamiltonian $\mathscr{H}'(\sigma)$ for the σ spins by performing an average over all the values of the $\bar{\sigma}$ spins:

$$\exp(-\mathscr{H}'(\sigma)) = \sum_{\{\bar{\sigma}\}} \exp(-\mathscr{H}(s)). \tag{2.11}$$

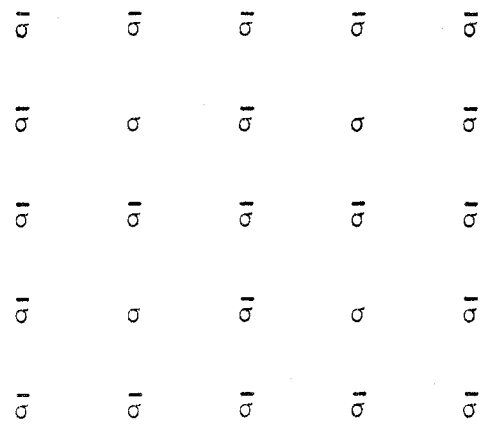

Figure 4. Spins σ and $\bar{\sigma}$ in a possible decimation scheme.

The value of the partition function of the new spins is equal to that of the partition function of the original spins:

$$\begin{aligned}\sum_{\{\sigma\}} \exp\left(-\mathscr{H}'(\sigma)\right) &= \sum_{\{\sigma\}} \sum_{\{\bar{\sigma}\}} \exp\left(-\mathscr{H}(s)\right) \\ &= \sum_{\{s\}} \exp\left(-\mathscr{H}(s)\right)\end{aligned} \tag{2.12}$$

since the spins $\{\sigma\}$ and $\{\bar{\sigma}\}$ together constitute the entire set $\{s\}$.

This formulation of a new effective Hamiltonian therefore realises the two basic principles of the renormalisation group method: thermodynamic quantities of the original system can be calculated from the new (e.g. the partition functions Z are the same!) and the new system is further from criticality (the ratio of correlation length to lattice spacing has been reduced by the factor $\frac{1}{2}$ for the case illustrated in figure 4). These properties are not self-contradictory: in $\mathscr{H}'$ there is a spin-independent constant term which builds up the partition function of the new to the value Z at each iteration.

We are faced now with the evaluation of the partial configuration sum (2.11). This cannot in general be done exactly, so we must resort to some approximation scheme. Now in this partial configuration sum, the σ spins are held fixed and a nearest-neighbour interaction $\sigma\bar{\sigma}$ actually looks like an external field term acting on the $\bar{\sigma}$ spin. Therefore this partial configuration sum can be understood as representing a thermodynamic system far from its critical point. This is the kind of physical argument raised to justify approximations in the calculation of the new effective Hamiltonian $\mathscr{H}'$.

In order to obtain a simple recursion formula to play with, we shall make an approximation for $\Sigma_{\{\bar{\sigma}\}}$ by expanding $\exp(-\mathscr{H})$ in $\mathscr{H}$, as in the series expansion (2.7). If $\mathscr{H}$ contains only nearest-neighbour interactions, the result may be written

$$\sum_{\{\bar{\sigma}\}} \exp\left(-\mathscr{H}(s)\right) = 2^{\bar{N}}\left[1 + N_b K^2/2! + K^2 \sum_{\langle\mu\nu\rangle} \sigma_\mu \sigma_\nu + O(K^4)\right] \tag{2.13}$$

where we have labelled the new sites by μ, ν, etc. If we identify this with the expansion of the exponential of the new Hamiltonian as in (2.9), we may write

$$-\mathscr{H}' = -F + K' \sum_{\langle\mu\nu\rangle} \sigma_\mu \sigma_\nu + O(K^4) \tag{2.14}$$

with

$$-F = \bar{N} \ln 2 + N_b K^2/2! + O(K^4) \tag{2.15}$$

$$K' = K^2 + O(K^4). \tag{2.16}$$

In these expressions, $\bar{N}$ is the number of $\bar{\sigma}$ sites, $2^{\bar{N}}$ is the number of terms in the sum $\Sigma_{\{\bar{\sigma}\}}$, and N_b is the number of bonds in the original lattice. The primary rule to remember in obtaining equation (2.13) is that any term in which the spin $\bar{\sigma}$ at any site appears only *once* gives zero when averaged on the values $\bar{\sigma} = \pm 1$. There is therefore no contribution from first order in $\mathscr{H}$, and from second order one must take the same bonds from each $\mathscr{H}$ (giving $N_b K^2/2!$) or adjacent bonds joining nearest-neighbour σ spins.

Let us now close our eyes, forget about the terms of order K^4, and show how the recursion formula

$$K' = K^2 \tag{2.17}$$

is used to obtain information on the phase transition.

The first remark to be made is that the recursion formula (2.17) has, in particular, the *fixed point* (others are discussed in §2.3.3)

$$K^* = 1. \tag{2.18}$$

If $K = 1$, then the new effective coupling K' has the *same* value 1. If K is just larger than 1, then K' is larger than K, i.e. further away from 1. Similarly, $K < 1$ implies $K' < K$. We say that the fixed point is *unstable* to a perturbation away from it (see figure 5).

Figure 5. Illustration of the flow from the recursion formula (2.17); successive dots indicate values of nearest-neighbour coupling after each iteration.

This behaviour must be compared with the property built into the RG: if we start with a system that is close to, but not at, its critical point, the new system is slightly further from its critical point. On this basis we must identify the value $K = 1$ as corresponding to the critical point of the system. Therefore we make the prediction

$$\frac{J}{kT_c} = 1. \tag{2.19}$$

This is to be compared with the result from Onsager's exact solution of this problem, $J/kT_c \simeq 0{\cdot}44$.

We can go further and ask what is the behaviour of the system close to its critical point. In order to study this we expand equation (2.17) in a Taylor series for K close to K^*. Simple algebra shows that

$$K' - K^* = 2(K - K^*) + O(K - K^*)^2. \tag{2.20}$$

For a system sufficiently close to its critical temperature, the terms of order $(K - K^*)^2$ are negligible. Now, we should compare this linearised equation with the fact that, *in units of respective lattice spacing*, the correlation length has been reduced by a factor of $\frac{1}{2}$, i.e.

$$\xi' = \tfrac{1}{2}\xi. \tag{2.21}$$

Equations (2.20) and (2.21) say that $(K-K^*)$ and ξ must be inversely related to one another; if we double one, we must halve the other, i.e.

$$\xi \propto (K-K^*)^{-1}. \tag{2.22}$$

Now if we look at what this relation implies for a system with a fixed J, with K varying as T is varied according to $K=J/kT$, we find

$$\xi(T) \propto (T-T_c)^{-1}. \tag{2.23}$$

Thus we see that the scaling phenomenon seen in equations (2.1) and (2.2) can be predicted in the RG approach. Moreover, the values of critical exponents are calculable; equations (2.2) and (2.23) imply

$$\nu = 1 \tag{2.24}$$

for the two-dimensional Ising model. Fortuitously, this is the exact result found in Onsager's solution of this model.

2.3.3. Further remarks. More insight can be gained from this simple example.

(*a*) There are also two other fixed points of the recursion formula (2.17), namely $K^*=0$ and $K^*=\infty$. The first corresponds to a system at infinite temperature, where the spins act essentially independently of one another because the coupling between them vanishes. The second corresponds to a system at $T=0$, where the effect of the coupling is so strong that the system sits in one of the aligned configurations of lowest energy. These fixed points are sometimes referred to as 'trivial' in the sense that they describe systems at $J=0$ or $T=0$. The critical fixed point $K^*=1$ is distinguished by the special property that the effective (non-trivial) coupling between the σ spins is the same as the coupling between the s spins, and so on at each iteration; this reflects the existence of cooperative fluctuations of spins of *all* length scales. More generally, we shall see in §3 how fixed-point Hamiltonians correspond to systems which are *scale*-invariant in a conventional sense; the physics they describe is unchanged if the lattice spacing is changed by a factor of 2, 4, 8, etc.

(*b*) What happens if we decimate, leaving every ninth spin? Then since the σ spins are now separated by three bonds, it takes third order in perturbation theory to produce a nearest-neighbour interaction. One readily finds

$$K' = K^3 + O(K^5).$$

Neglecting order K^5 gives non-trivial fixed points

$$K^* = \pm 1.$$

The fixed point $K^*=+1$ can be analysed as before: expanding about it, we find

$$K'-K^* = 3(K-K^*) + O(K-K^*)^2.$$

This must now be compared with a dimensionless correlation length which has decreased by a factor of $\frac{1}{3}$. Therefore we again obtain $\nu=1$.

The fixed point $K^*=-1$ is new and capable of physical interpretation. It just corresponds to a fixed point for negative K, i.e. an antiferromagnetic system (see the remarks after equation (2.3)). It appears only when the new spins are separated by an odd number of old lattice spacings. The ordered state of the antiferromagnetic

system is shown in figure 3, and will be reflected in the form of the fluctuations which are the precursors of it; only when the lattice spacing changes by an odd integer does such an antiferromagnetic fluctuation still look antiferromagnetic.

There is a very important moral to be drawn from this simple example. The RG can be applied with confidence only when the nature of the cooperative fluctuations is recognised; it must successively eliminate such fluctuations of larger and larger range.

(*c*) This section would be incomplete without a warning; the simple decimation scheme outlined above has many serious defects. We obtain the recursion formula valid for small K and use it at $K \sim 1$; when higher orders in K are included there may exist several fixed points which are impossible to interpret. It would imply a finite transition temperature in a one-dimensional system, although it is known that none exists. In fact, for $d=1$, an *exact* recursion relation may be derived for the decimation scheme in which only one out of every b spins is kept. The result is

$$\tanh K' = (\tanh K)^b$$

which has fixed points only at $K=0$ and ∞.

Also, decimation in this naive fashion is correct only for fixed points describing systems whose critical exponent η is exactly $2-d$. This condition is not satisfied by most systems. In §4.2, we discuss the origin of this difficulty and review several methods avoiding it.

2.4. Summary

In a magnetic system all fluctuations with a range between the lattice spacing and the correlation length are important. As the system approaches its critical point, the correlation length becomes macroscopic. The RG approach to this many-body problem is to set up a transformation which is applied many times and successively removes the effect of fluctuations of larger and larger size at the expense of renormalising the effective interactions. Critical behaviour of the system is governed by fixed point (s) of the RG transformation, and the phase transition temperature, scaling behaviour and values of critical exponents can all be predicted from it. These ideas may be applicable in any circumstance where the interactions are formulated at one length scale and cooperative fluctuations are important over a whole range of length scales, although the nature of the fluctuations must be recognised.

We end this section by listing a few general references. For more complete discussions on the phenomenology of scaling in critical phenomena, see Stanley (1971), Fisher (1967a) and Kadanoff *et al* (1967); the liquid–gas system is discussed in some detail by Sengers and Sengers (1976). Details on exactly soluble models can be found in McCoy and Wu (1973), Baxter (1972) and Thomson (1972). Series expansion approximation schemes are reviewed in Domb and Green (ed) (1974) and Monte Carlo methods by Binder (1976). The idea of setting up a transformation by blocking spins on a lattice is due to Kadanoff (1966). Exact RG transformations for lattice models in one dimension are discussed by Nelson and Fisher (1975) and Balian and Toulouse (1974); realistic calculations with the decimation method are described in Wilson (1975). General reviews of the RG method are given by Ma (1973b, 1976b), Wilson and Kogut (1974), Fisher (1974a), Toulouse and Pfeuty (1975) and others; some will be listed later as appropriate.

We mention also that the RG has found applications in many other branches of physics. In fact, it was originally formulated in relativistic quantum field theories of elementary particles by Gell-Mann and Low (1954) and Stueckelberg and Petermann (1953): for modern reviews, see e.g. Politzer (1974) and Macfarlane (1977). The application to the Kondo problem is discussed by Wilson (1975) and renormalisation in turbulence by Kraichnan (1975). We give these only to indicate the breadth of applicability of the RG ideas; more references to applications are given in §6.

3. The renormalisation group—general formalism

In the previous section, we saw how the critical temperature and a critical exponent can be obtained in a simple RG calculation. In this section we shall look at the general RG formalism and discuss how other physical properties are obtained from it. A summary of the techniques for actually calculating the RG transformation functions in lattice models is given in the next section.

3.1. Critical exponents and universality

It is clear that if we calculate higher orders in K in the decimation scheme described in the previous section we will produce new types of coupling. If we generate such coupling at one iteration, it appears in the starting Hamiltonian for the next iteration; to be consistent therefore we must include it in the starting Hamiltonian. In fact, every RG calculation on a lattice with any pretence to accuracy involves several coupling constants. We write

$$\mathscr{H} = -\sum_i K_i S_i \tag{3.1}$$

where S_i denotes a particular interaction type. For example, the nearest-neighbour interaction, S_1 say, is

$$S_1 = \sum_{\langle mn \rangle} s_m s_n$$

as in equation (2.3). Then the typical recursion formula has the general form of equation (2.9)

$$K_i' = f_i(\boldsymbol{K}) \qquad i = 1, 2, 3, \ldots . \tag{3.2}$$

The complications of including several types of interaction have some compensation however; we shall see how the feature of universality can arise.

3.1.1. Mathematical formalism. Fixed points K_i^* of the recursion formula (3.2)

$$K_i^* = f_i(\boldsymbol{K}^*) \tag{3.3}$$

play a very important role in the general case, as they did in the analysis of equation (2.17). The solutions K^* of this equation may have a very rich structure; there may be isolated roots of (3.3), or families of solutions parametrised by a set of continuous variables, or no solutions at all. The existence of *some* fixed points seems to be an essential feature for obtaining a description of phase transitions. When we plot the solution of equation (3.3) in the space of coupling constants, we shall assume in this section that they form an isolated set of points. From the example of the

previous section, we expect to distinguish at least three classes of fixed points—a high-temperature fixed point, in which all the K vanish, a set of low-temperature fixed points, with some K infinite, and a set of 'critical' fixed points, with non-zero finite K.

Given a fixed point, we can ask how the recursion formula behaves for points close to K^*. This is obtained by expanding $f_i(\boldsymbol{K})$ in a Taylor series expansion about K^*, as in equation (2.20). We have

$$K_i' = f_i(\boldsymbol{K}^*) + \frac{\partial f_i}{\partial K_j}(\boldsymbol{K}^*).(K-K^*)_j + \mathrm{O}(K-K^*)^2$$

or, using (3.2) and neglecting $\mathrm{O}(K-K^*)^2$,

$$(K'-K^*)_i = M_{ij}(K-K^*)_j \qquad (3.4(a))$$

where

$$M_{ij} = \frac{\partial f_i}{\partial K_j}(\boldsymbol{K}^*). \qquad (3.4(b))$$

Note that we use the repeated index summation convention, unless otherwise stated.

A linear problem like equation (3.4) is always tractable provided we can diagonalise the matrix M_{ij}. The eigenvalues λ_a and (left) eigenvectors $v_j^{(a)}$ of M_{ij} are defined by

$$\sum_j v_j^{(a)} M_{ji} = \lambda_a v_i^{(a)} \qquad a = 1, 2, \ldots . \qquad (3.5)$$

If we define 'normal' coordinates q_a by

$$q_a = \sum_i v_i^{(a)}(K-K^*)_i \qquad (3.6)$$

we find the *decoupled* equations:

$$q_a' = \lambda_a q_a. \qquad (3.7)$$

Note that the corresponding right eigenvectors $u_i^{(a)}$ enable (3.6) to be inverted for $(K-K^*)_i$ in terms of q_a.

There are two assumptions implicit here. Although M is a real matrix it is in general not symmetric (hence the distinction between left and right eigenvectors). Therefore it may not be diagonalisable, but may only be reducible to Jordan canonical form. Even if it is diagonalisable, it need not have real eigenvalues. Since in almost all practical calculations known to us M is diagonalisable with real eigenvalues, we shall assume this in the general formalism.

It should now be clear what happens.

(i) The components q_a for which $\lambda_a > 1$ grow; $q_a' > q_a$ and the new parameters $\boldsymbol{K}'$ are further from $\boldsymbol{K}^*$ than the original ones $\boldsymbol{K}$. The interaction associated with this perturbation, $\Sigma_i u_i^{(a)} S_i$, is said to be *relevant* for the fixed point $\boldsymbol{K}^*$.

(ii) The components q_a for which $\lambda_a < 1$ decrease; $q_a' < q_a$ and the new parameters $\boldsymbol{K}'$ are closer to $\boldsymbol{K}^*$ than the original ones $\boldsymbol{K}$. The interactions associated with such perturbations are said to be *irrelevant* for the fixed point $\boldsymbol{K}^*$.

(iii) Eigenvectors for which $\lambda_a = 1$ are said to be *marginal*; in the linearised equation (3.7) they neither grow nor decrease. Only by including the higher-order terms in q can we determine what happens to such perturbations.

A typical picture of what happens to the parameters under the RG is shown in figure 6, for the case where there are three couplings and the fixed point $\boldsymbol{K}^*$ has

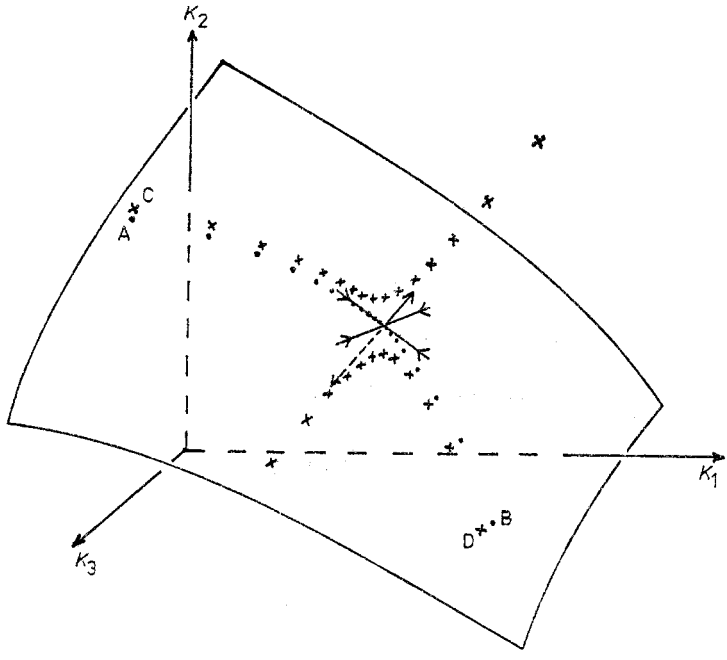

Figure 6. Schematic illustration of flow near a fixed point. Points A and B start in the critical sheet and iterate (dots) into the fixed point. Point C (D) starts just 'above' ('below') the critical sheet, then moves past the fixed point and away along the (opposite) relevant eigendirection.

one relevant and two irrelevant eigenvalues. The points give the values of the parameters $\{K_i^{(l)}\}$ as the RG transformation is iterated ($l = 1, 2, 3, \ldots$) starting from various initial points A, B, C and D. The stable manifold of the fixed point is the set of points which are attracted to the fixed point under repeated iteration of the RG; it is a two-dimensional manifold in this case, spanned by the two irrelevant eigenvectors for small q, and differing from the plane of these vectors by terms of second order in q. Points A and B lie in the stable manifold; points C and D are just off the stable manifold and the Hamiltonians (sets of couplings) obtained from them under RG transformation come close to the fixed point but end up being driven away from it along the relevant eigenvector.

An implicit assumption in all of the above is that the transformation functions $f_i(\boldsymbol{K})$ are regular; the derivatives (3.4(*b*)) exist and, as in figure 6, points which start close together are still close after one iteration. No singularities appear in these functions and they can be calculated using approximation schemes (see remarks in §2.3).

3.1.2. Physical interpretation. In addition to the structure of fixed points and their stable manifolds which can be erected in the space of coupling constants using the recursion formula (3.2), one can also define surfaces corresponding to systems of a given dimensionless correlation length (ratio of correlation length to lattice spacing). In particular, one can define the *critical surface* corresponding to systems *at* their critical point, with $\xi = \infty$. For an ordinary magnetic transition, we must control two parameters, the external field (equal to 0) and the temperature (equal to T_c)

to be at a system's critical point. Therefore for an ordinary magnetic system the critical surface has two dimensions less than the space of all the coupling constants we consider.

In the simple example we considered in the previous section, there was no external field and we had only one coupling constant. The critical sheet in this example is then just a point—the value of K at which the system goes through a phase transition. In that example there is a very close connection between the fixed point and the critical point—they are identical. In the general case we consider in this subsection this connection is extended: the critical surface corresponds to the stable manifold of the fixed point describing the magnetic phase transition.

In order to appreciate this point, the reader is reminded that we know qualitatively what happens when we do an RG transformation on a system which is *close* to criticality; the transformed system has fewer degrees of freedom in a correlation length than the original system and hence is not so close to criticality. If the original system is actually *at* its critical point (the correlation length is infinite), then the new system is also at its critical point, because the number of new spins in a correlation length is still infinite.

Therefore, trajectories which start in the critical surface remain in it. Trajectories which start off the critical surface move away from it. Hence any fixed point *in* the critical surface must be *unstable* to perturbations out of the critical surface. In other words, the RG transformation is constructed so that the critical fixed point is unstable to perturbations involving precisely those interactions which must be controlled to be at a critical point.

Of course, there may exist other fixed points in the critical surface (we shall discuss their significance later). However, if there does exist a fixed point which is stable to all perturbations within the critical surface and unstable only to perturbations out of the critical surface, then it will control the behaviour of a 'generic' (i.e. typical) magnetic transition. For its stable manifold will coincide with the critical surface. Then all systems *at* their critical point will move in to $\boldsymbol{K}^*$ under the RG (cf points A and B in figure 6). All systems close to their critical point (cf C and D in figure 6) will pass close to $\boldsymbol{K}^*$ and their growth away from $\boldsymbol{K}^*$ will be determined by the same relevant eigenvalues λ_a of equation (3.7) (qualifications due to other fixed points in the critical surface are discussed later).

The conclusion of this discussion is that if we can find a fixed point which is only twice unstable, to perturbations in T away from T_c and H away from zero, then we shall have *universal* critical behaviour. In fact, the critical exponents associated with H and $(T-T_c)$ are obtained from the eigenvalues of the corresponding relevant interactions; the eigenvalues of the irrelevant operators ($\lambda_a < 1$) are associated with correction-to-scaling exponents.

The proof of these last two statements requires only a refinement of the arguments following equation (2.20). Firstly, consider a system which differs from $\boldsymbol{K}^*$ by a perturbation q_1 along the unstable eigenvector $u_i^{(1)}$ corresponding to perturbations of T away from T_c. Since the couplings $\boldsymbol{K}$ are analytic in T ($T \neq 0$) we can take

$$q_1 \propto (T - T_c) \tag{3.8}$$

up to negligibly small terms of order $(T-T_c)^2$. Then the recursion formula $q_1' = \lambda_1 q_1$ is equivalent to

$$(T - T_c)' = \lambda_1 (T - T_c). \tag{3.9}$$

If the RG corresponds to a change in length scale by a factor b at each iteration, then the dimensionless correlation length transforms according to

$$\xi' = b^{-1}\xi. \tag{3.10}$$

Combining equations (3.9) and (3.10) we obtain

$$\xi \propto (T - T_c)^{-\nu} \qquad (H = 0) \tag{3.11}$$

with

$$\nu = \ln b / \ln \lambda_1. \tag{3.12}$$

Similarly, if the eigenvector corresponding to an external field obeys

$$h' = \lambda_2 h \tag{3.13}$$

then at $T = T_c$ the correlation length ξ tends to infinity as

$$\xi \propto h^{-\ln b/\ln \lambda_2}. \tag{3.14}$$

Another illuminating way of obtaining these results is as follows. After l iterations, we have

$$\xi^{(l)} = b^{-l}\xi \qquad q_1{}^{(l)} = (\lambda_1)^l q_1. \tag{3.15}$$

According to the principles outlined in the previous section these equations are to be iterated a suitable number, L say, of times until the new system is far from criticality. This is reached when

$$\xi^{(L)} = A \qquad q_1{}^{(L)} = B \tag{3.16}$$

where A and B are both of order 1; the correlation length ceases to be large when the perturbation out of the critical surface ceases to be small. It is a straightforward matter to eliminate L in equations (3.15) and (3.16) to obtain

$$\xi = AB^{\nu} q_1^{-\nu} \tag{3.17}$$

with ν as in (3.12).

Secondly, suppose that we have an irrelevant interaction

$$q_3' = \lambda_3 q_3 \qquad (\lambda_3 < 1) \tag{3.18}$$

in addition to a relevant one, q_1 say, which again gives a measure of $T - T_c$. This corresponds to a system C or D in figure 6. After the L iterations, q_3 will be very small:

$$q_3{}^{(L)} = (\lambda_3)^L q_3.$$

Precisely how small $q_3{}^{(L)}$ is, is determined by how large L is, i.e. how close T is to T_c initially. By eliminating L using (3.15) and (3.16), one finds

$$q_3{}^{(L)} = q_3[(T - T_c)/B]^{-\ln \lambda_3/\ln \lambda_1}. \tag{3.19}$$

Since the system after L iterations has no critical fluctuations, we can then perturb in the variable $q_3{}^{(L)}$. This perturbation produces corrections to the case $q_3 = 0$ proportional to

$$q_3 (T - T_c)^{-\ln \lambda_3/\ln \lambda_1}. \tag{3.20}$$

Note that such terms are arbitrarily small, even if q_3 is not small, provided that T is close enough to T_c. They provide corrections to the leading singularities such as (3.11), governed by the correction-to-scaling exponent $-\ln \lambda_3/\ln \lambda_1$ (> 0). These

corrections are confluent singularities—they correspond to weaker branch cuts on top of leading singularities in $(T - T_c)$.

We end this subsection by summarising these results. In order to obtain the maximum universality in a magnetic transition, we look for a fixed point which is unstable only to perturbations corresponding to H and $T - T_c$. The stable manifold of this fixed point then corresponds to the critical surface. All systems close to this stable manifold exhibit universal critical behaviour. Critical exponents are given by eigenvalues of the transformation linearised about the fixed point. Corrections to the leading universal critical behaviour are governed by the eigenvalues of irrelevant interactions.

A final remark: the linearised recursion formulae are valid strictly only very close to the fixed point. The iteration to L brings them outside this region but if we start with q arbitrarily small, then the vast majority of iterations occur close to $\boldsymbol{K}^*$ and the linearisation becomes arbitrarily good. These comments may be formulated quantitatively by constructing so-called non-linear scaling fields Q, defined by

$$\begin{aligned} Q_a &= q_a + \mathrm{O}(q^2) \\ Q_a' &= \lambda_a Q_a \end{aligned} \tag{3.21}$$

where this second equation is satisfied for all Q_a (not necessarily small). In the space of K the Q are curvilinear coordinates and $Q_1 = 0 = Q_2$ gives the critical sheet. The critical value for the nearest-neighbour coupling K in the Ising model is given by the intersection of the critical sheet with the K axis. If the fixed point is not too far from this axis, an approximate value for the critical K is obtained from the intersection of the tangent plane at the fixed point (spanned by the $u_i^{(a)}$ of irrelevant interactions) with the K axis (see Niemeijer and van Leeuwen 1976, Dalton and Wood 1969). In terms of the curvilinear coordinates the corrections to scaling take the form:

$$Q_3 (T - T_c)^{-\ln \lambda_3 / \ln \lambda_1}$$

these are small with no restriction on the (initial) size of Q_3 provided $(T - T_c)$ is small enough.

The general analysis of systems with several coupling constants is given by Wegner (1972a). Non-linear scaling fields are discussed by Riedel and Wegner (1974). Global features associated with the non-linear fields are discussed by Nicoll *et al* (1975). Further discussion of these topics can be found in several reviews, e.g. Fisher (1974a), Niemeijer and van Leeuwen (1976) and Wegner (1976).

3.2. Scaling in free energy, equation of state and correlation functions

Having had a qualitative understanding of the relationship between fixed points (and their neighbourhoods) of RG transformations and the important features (universality, singularities, critical exponents) of a second-order transition, we will examine how these features arise in thermodynamic quantities in a more detailed, quantitative fashion. We follow the treatment reviewed in Niemeijer and van Leeuwen (1976).

First, let us present the medium for discussing more than one critical exponent simultaneously, i.e. functions which 'scale' in several variables: the class of generalised

homogeneous functions. They satisfy

$$F(x_1, \ldots, x_n) = p^{y_0} F(p^{y_1} x_1, \ldots, p^{y_n} x_n) \tag{3.22}$$

with p being a positive parameter and y_i being arbitrary (real) numbers. Although y_0 can be chosen to be unity without loss of generality, it is kept for the convenience of identifying p with the scale change, b, in an RG transformation later. An introduction to generalised homogeneous functions is given in Stanley (1971, chap 11).

So far we have considered (3.22) only for functions of a single variable. For example, if ξ, as a function of $t \equiv T - T_c$, satisfies

$$\xi(t) = b\xi(\lambda_1 t)$$

then ξ is necessarily (up to oscillations) proportional to t^{-1/y_1}, with

$$y_1 = \ln {}_b\lambda_1 \equiv \ln \lambda_1 / \ln b.$$

This subsection is devoted to showing how the 'most singular' part of thermodynamic functions can be expressed as generalised homogeneous functions, extracting leading corrections to these parts, and deriving the scaling laws (relationships among critical exponents). The phenomenon of universality will be examined in a little more detail.

Unless it is explicitly stated otherwise, we adopt the convention of absorbing a factor of $1/kT$ into all Hamiltonians and energies so that they are dimensionless quantities.

3.2.1. Free energy. Given a Hamiltonian $\mathscr{H}(\boldsymbol{K})$, characterised by a set of couplings $\{\boldsymbol{K}\}$, the Gibbs free energy $G(\boldsymbol{K})$ can be calculated via the partition function (equation (2.5)):

$$G(\boldsymbol{K}) = -\ln \sum_{\{s\}} \exp(-\mathscr{H}). \tag{3.23}$$

Similarly, we have G' for the transformed system, characterised by $\mathscr{H}'(\boldsymbol{K}')$:

$$G'(\boldsymbol{K}') = -\ln \sum_{\{\sigma\}} \exp(-\mathscr{H}'). \tag{3.24}$$

Since $\mathscr{H}'$ is defined so that the partition function remains the same, we have

$$G = G'. \tag{3.25}$$

This subsection is devoted to extracting information out of this relation.

In general, there is a part of $\mathscr{H}$ which plays an easily tractable role in the RG transformations: the spin-independent part (e.g. F of $\mathscr{H}'$ in equation (2.14)). Denoting this term (per site) by K_0, we write

$$\mathscr{H} = NK_0 + \mathscr{H}_R \tag{3.26}$$

where $\mathscr{H}_R$ is the spin-dependent part, obeying $\Sigma_{\{s\}} \mathscr{H}_R = 0$. The term NK_0 adds nothing but a trivial constant to the free energy, so without loss of generality we assume that K_0 is zero. $\mathscr{H}'$ still has a spin-independent part:

$$\mathscr{H}' = N'K_0' + \mathscr{H}_R'$$

where K_0' is determined by the couplings $\{\boldsymbol{K}\}$ of $\mathscr{H}$; we write

$$K_0' = f_0(\boldsymbol{K}) b^d. \tag{3.27}$$

We assume that f_0, like f_i $(i > 1)$, is a regular function of $\boldsymbol{K}$.

Next let us assume the existence of a thermodynamic limit so that the Gibbs free energy *per site* is well defined:

$$g(\boldsymbol{K}) = -\frac{1}{N} \ln \sum_{\{s\}} \exp(-\mathscr{H}). \tag{3.28}$$

Note that the function g is defined here for a Hamiltonian with no spin-independent part. Equations (3.23)–(3.25) give

$$\begin{aligned} g(\boldsymbol{K}) &= -\frac{1}{N} \ln \sum_{\{\sigma\}} \exp(-\mathscr{H}) \\ &= -\frac{1}{N} \ln \sum_{\{\sigma\}} \exp(-N'K_0' - \mathscr{H}_R') \\ &= \frac{N'}{N} K_0' - \frac{1}{N} \ln \sum_{\{\sigma\}} \exp(-\mathscr{H}_R'). \end{aligned} \tag{3.29}$$

From the definition of g in equation (3.28), we see that the last term is $(N'/N)g(\boldsymbol{K}')$. Since $N'/N = b^d$, we arrive at an important equation for g:

$$g(\boldsymbol{K}) = f_0(\boldsymbol{K}) + b^{-d} g(\boldsymbol{K}'). \tag{3.30}$$

In a domain around some fixed point where (3.21) is well defined, we can eliminate the $\boldsymbol{K}$ in favour of the scaling fields Q_a. Denoting the free energy per site again by $g(Q_a)$, we have from (3.21)

$$g(Q_a) = f_0(Q_a) + b^{-d} g(b^{y_a} Q_a) \tag{3.31}$$

where

$$y_a = \ln {}_b\lambda_a. \tag{3.32}$$

Except for the inhomogeneous term (f_0), g is a generalised homogeneous function of the scaling fields Q (cf equation (3.22)).

As an appetiser, let us look at the consequences of (3.31). Suppose g has a part, g_s, which is a singular function of, say, Q_1 at the origin. By assumption f_0 is regular. So g_s satisfies (3.31) without the f_0 term. Following the analysis for ξ at the beginning of this subsection, we arrive at

$$g_s \propto |Q_1|^{d/y_1}. \tag{3.33}$$

If Q_1 corresponds to perturbations away from T_c, we have $Q_1 = q_1 + O(q_1^2) \propto (T - T_c) + O(T - T_c)^2$, so one predicts singularities of the form $|T - T_c|^{d/y_1}$.

3.2.1.1. The case of one scaling field. To derive the form (3.33) as well as an expression for the coefficient of proportionality let us consider the simplest case of one scaling field Q (the index 1 is now dropped for clarity). The solution to (3.31) may be written down by iteration provided that the remainder tends to zero and the series converges: if

$$\lim_{L \to \infty} b^{-Ld} g(\lambda^L Q) = 0 \tag{3.34}$$

then we have

$$g(Q) = \sum_{l=0}^{\infty} b^{-ld} f_0(\lambda^l Q). \tag{3.35}$$

The condition (3.34) is not automatic and must be checked in all cases. Since f_0 is regular by assumption (and in practice), we have

$$f_0(Q) = \sum_{n=0}^{\infty} C_n Q^n. \tag{3.36}$$

If λ is less than unity (Q irrelevant), then $\lambda^n b^{-d} < 1$ for all n, so that on substituting (3.36) into (3.35) the order of summations may be interchanged and the l summation may be done. The result is

$$g(Q) = \sum_{n=0}^{\infty} \frac{C_n}{1-\lambda^n b^{-d}} Q^n \equiv g_r(Q) \tag{3.37}$$

which is a regular function of Q, the series having a radius of convergence λ times that of (3.36).

For $\lambda > 1$ (Q relevant), $\lambda^n b^{-d} > 1$ for all large n and the order of summations may *not* be interchanged. We can still define $g_r(Q)$ by the series (3.37). The difference between $g(Q)$ and $g_r(Q)$ is obtained as follows.

Suppose $\lambda^n \neq b^d$ for any integer n. Let m be the largest value of n such that $\lambda^n b^{-d} < 1$, i.e.

$$\lambda^m b^{-d} < 1 < \lambda^{m+1} b^{-d}. \tag{3.38}$$

Now split f_0 into the sum of $f_<$ and $f_>$, where

$$f_>(Q) \equiv \sum_{n=m+1}^{\infty} C_n Q^n \tag{3.39(a)}$$

and

$$f_< = f_0 - f_> \tag{3.39(b)}$$

so that

$$g - g_r = \sum_{l=0}^{\infty} b^{-ld} f_>(\lambda^l Q) - \sum_{n=m+1}^{\infty} \frac{C_n}{1-\lambda^n b^{-d}} Q^n. \tag{3.40}$$

For the terms in the second sum in this equation, we write

$$\frac{-1}{1-\lambda^n b^{-d}} = \sum_{l=-\infty}^{-1} (\lambda^n b^{-d})^l$$

so that (3.40) becomes

$$g = g_r + g_s \tag{3.41}$$

with the singular part g_s given by

$$g_s(Q) = \sum_{-\infty}^{\infty} b^{-ld} f_>(\lambda^l Q). \tag{3.42}$$

Since it is a sum over *all* integers l, g_s is a generalised homogeneous function, obeying

$$g_s(Q) = b^{-d} g_s(b^y Q)$$

with $y = \ln_b \lambda$. Hence it is of the form

$$g_s(Q) = |Q|^{d/y} \left(\sum_n A_n^{\pm} |Q|^{2\pi i n / \ln \lambda} \right) \tag{3.43}$$

where $A_n^{\pm}$ are amplitudes of the nth oscillatory term and $\pm$ refers to the sign of Q.

The possible existence of oscillatory terms was noted by Wilson (1971a, b); see also Nauenberg (1975). Although we cannot eliminate them *a priori* if only one RG transformation is used, they are absent in formulations of the RG in which p in (3.22) can vary continuously, and even in the present formulation they are in practice negligible. We examine therefore only the $n=0$ term. Then g_s reduces to

$$g_s(Q)=A^{\pm}|Q|^{d/y} \qquad Q\gtrless 0 \tag{3.44}$$

with

$$A^{\pm}=\frac{1}{\ln\lambda}\int_0^{\infty} d\tau\, \tau^{-(1+d/y)} f_{>}(\pm\tau). \tag{3.45}$$

To obtain (3.45), perform an inverse Fourier transformation in $\ln Q$ on (3.43), change integration variable to $\tau=\lambda^l|Q|$ in the lth term, and note

$$\sum_{l=-\infty}^{\infty}\int_{I(l)} d\tau=\int_0^{\infty} d\tau \qquad (I(l)\equiv[\lambda^l,\lambda^{l+1}]).$$

The integral in (3.45) is convergent at $\tau=0$ from the definition of $f_>$ in (3.38) and (3.39). The behaviour as $\tau\to\infty$ is governed by the behaviour of $f_0(\lambda^l Q)$ as $\lambda^l Q\to\infty$, i.e. $l\to\infty$. After a large number of iterations, a system not at its critical point tends to a fixed point at $T=0$ or $T=\infty$. In each of these limits f_0 will have same definite value and sign. The convergence of the monotonic sum $\Sigma\, b^{-ld} f_0(\lambda^l Q)$ (equations (3.34) and (3.35)) implies the convergence of the integral $\int_B^{\infty} d\tau\, \tau^{-(1+d/y)} f_0(\tau)$. Since $\int_B^{\infty} d\tau\, \tau^{-(1+d/y)} f_<(\tau)$ converges, according to (3.38) and (3.39), then (3.45) also converges at infinity.

The case $\lambda^k=b^d$ for some integer k has to be treated separately because (3.37) becomes ill-defined. To handle this situation let us suppose we have $\lambda-\epsilon$ as an eigenvalue ($\lambda^k=b^d$ with k integer) and then take the limit $\epsilon\to 0_+$. The $O(1/\epsilon)$ term in g_r is now cancelled by a similar one in g_s. First, since $y=\ln_b(\lambda-\epsilon)$, we have

$$\frac{d}{y}=k+\frac{k\epsilon}{\lambda\ln\lambda}+O(\epsilon^2). \tag{3.46}$$

Next, the behaviour of $f_>(\tau)$ at large τ will be $-C_k\tau^k$ since k is now the maximum power in $f_<$ (cf (3.38)). So let us define an amplitude which will be finite at $\epsilon=0$ by (for $Q>0$):

$$A\equiv \lim_{\epsilon\to 0_+}\frac{1}{\ln(\lambda-\epsilon)}\int_0^{\infty} d\tau\, \tau^{-(1+(d/y))}[f_>(\tau)+\theta(\tau-1)C_k\tau^k] \tag{3.47}$$

where $\theta(x)=1(0)$ for $x>0$ ($x<0$). Now

$$g_s=AQ^k+O(\epsilon)+Q^k\left(1+\frac{k\epsilon}{\lambda\ln\lambda}\ln Q+O(\epsilon^2)\right)\frac{C_k}{\ln(\lambda-\epsilon)(k-d/y)}. \tag{3.48}$$

The denominator of the last factor is, in fact, $-(k\epsilon/\lambda)(1+\epsilon/2\lambda)$. Meanwhile, in g_r the $n=k$ term is just

$$C_kQ^k[1-(\lambda-\epsilon)^k b^{-d}]^{-1}=C_kQ^k\left(\frac{\lambda}{k\epsilon}\right)\left(1+\tfrac{1}{2}(k-1)\frac{\epsilon}{\lambda}+O(\epsilon^2)\right) \tag{3.49}$$

so that $g = g_r + g_s$ is finite in the $\epsilon \to 0_+$ limit, i.e.

$$g = \sum_{n \neq k} \frac{C_n Q^n}{1 - \lambda^n b^{-d}} + (\tfrac{1}{2} C_k + A) Q^k - \frac{C_k}{\ln \lambda} Q^k \ln Q. \tag{3.50}$$

Since k is an integer, we remark that the power singularity in (3.45) has become a logarithmic one here.

Finally, if $\lambda = 1$ (marginal), then higher orders in q are required to determine stability of the fixed point. Additional logarithmic factors violating scaling appear in this case (see, for example, Wegner 1972a, Wilson 1975). In numerical calculations on a lattice this case is rather rare, but see §§6.6 and 6.8.

3.2.1.2. The case of several scaling fields. The generalisation to the case of many scaling fields is immediate. Being regular f_0 may be written as

$$f_0 = \sum_{n_1, n_2, \ldots} C_{n_1, n_2, \ldots} \prod_a Q_a^{n_a} \tag{3.51}$$

and

$$g_r = \sum C_{n_1, n_2, \ldots} \left(1 - b^{-d} \prod_a \lambda_a^{n_a}\right)^{-1} \prod_a Q_a^{n_a} \tag{3.52}$$

with

$$g_s = \sum_{l=-\infty}^{\infty} b^{-dl} f_>(\lambda_1^l Q_1, \lambda_2^l Q_2, \ldots) \tag{3.53}$$

where $f_>$ is the sum (3.51) over only those n for which

$$\prod_a \lambda_a^{n_a} > b^d \tag{3.54}$$

in generalisation to (3.39(a)). Now, g_s satisfies (3.22) with

$$p = b \qquad y_0 = d \qquad y_a = \ln {}_b\lambda_a \qquad x_a = Q_a.$$

As before, in order for g_s to exist, there must be at least one relevant variable, Q_1 say (otherwise no set of n_a will satisfy (3.54)). To put g_s into the form (3.44), we will need an amplitude which is independent of the scale transformation yet dependent on the variables $Q_2, \ldots$. This property can be easily realised by first defining the scaling variables—ones that are independent of scaling:

$$V_a \equiv Q_a |Q_1|^{-y_a/y_1} \qquad a \geqslant 2. \tag{3.55}$$

To obtain the desired amplitude (a generalisation of equation (3.45)) we use these variables in dealing with the integrals over the intervals $[\lambda_1, \lambda_1^{l+1}]$. The integrands are, in fact, independent of l, since

$$\lambda^l{}_a Q_a = [\lambda_1^{y_a/y_1}]^l V_a |Q_1|^{y_a/y_1} = [\lambda_1^l |Q_1|]^{y_a/y_1} V_a = V_a \tau^{y_a/y_1}.$$

So we finally find that

$$A_1^{\pm}(V_2, \ldots) = \frac{1}{\ln \lambda_1} \int_0^{\infty} \mathrm{d}\tau \, \tau^{-(1+(d/y_1))} f_>(\pm \tau, V_2 \tau^{y_2/y_1}, \ldots) \tag{3.56}$$

is indeed scale-invariant, being a function only of the V. The subscript 1 on the amplitude reminds us that Q_1 out of all the relevant variables was the one chosen for casting g_s into the form of

$$g_s(Q_1, \ldots) = |Q_1|^{d/y_1} A_1^{\pm}(V_2, \ldots). \tag{3.57}$$

The phenomenon of universality, considered intuitively in the previous subsection, can now be derived. Suppose we have a fixed point with only one relevant variable (i.e. $y_a < 0$ for $a \geqslant 2$). Then as we approach the subspace of couplings defined by $Q_1 = 0$, V_a becomes as small as we please, for any finite Q_a ($a \geqslant 2$). $A_1^{\pm}$ approaches $A^{\pm}$ considered before, which is independent of the set $\{Q_a\}$. In other words, we have a universal behaviour when Q_1 is small.

Further, according to the definition of $f_>$ using (3.54), the integral in (3.56) remains convergent at $\tau = 0$ and $\tau = \infty$ when f is expanded in powers of $V_2, V_3, \ldots$. So $A_1^{\pm}$ is expandable in a Taylor series about $Q_a = 0$:

$$A_1^{\pm}(V_2, \ldots) = A_1^{\pm}(0) + V_2 A_{1;\,2}^{\pm} + \ldots \tag{3.58}$$

where

$$A_{1;\,a}^{\pm} = \frac{\partial}{\partial V_a} A_1^{\pm}|_{\{V\}=0}. \tag{3.59}$$

This expansion produces corrections to the most singular behaviour of g_s:

$$g_s(Q_1, \ldots) = A_1^{\pm}(0)\,|Q_1|^{d/y_1} + Q_2\,|Q_1|^{(d-y_2)/y_1} A_{1;\,2}^{\pm} + \ldots. \tag{3.60}$$

(Remember $y_2, \ldots < 0$.) As criticality is approached, the most important correction comes from the term with largest y_a, no matter how small (but not equal to zero) the corresponding Q_a is.

If $p(>1)$ relevant variables are present, then criticality is the subspace with $\{Q_1, \ldots, Q_p\} = 0$ and the 'critical region' is a region in which they are all small. Thus, we must keep the variables $V_2, \ldots, V_p$ (if we choose to extract Q_1) in order to describe the behaviour of g in this p-dimensional (critical) region. Universality is expressed by the fact that in the critical region

$$A_1^{\pm}(V_1, V_2, \ldots) \to A_1^{\pm}(V_1, V_2, \ldots, V_p, 0, 0, \ldots) \tag{3.61}$$

which is independent of the parameters $Q_{p+1}, \ldots$ and universal up to the choice of units for $Q_1, \ldots, Q_p$.

3.2.1.3. Scaling laws and equation of state. As an example, we consider g as a function of the relevant variables H and t, with eigenvalues b^{y_H} and b^{y_t}. Of course, H and t are not necessarily the non-linear scaling fields Q, but by symmetry under $H \to -H$, they may be taken as the linear scaling fields q. Since in the critical region we can neglect terms of order q^2 in $Q = q + O(q^2)$, we write the singular part of g as

$$g_s(H, t) = A_h^{\pm}(t\,|H|^{-1/\Delta})\,|H|^{d/y_H} \tag{3.62(a)}$$

$$= A_t^{\pm}(H\,|t|^{-\Delta})\,|t|^{d/y_t} \tag{3.62(b)}$$

where

$$\Delta = y_H/y_t.$$

To obtain the exponent β, we note that the spontaneous magnetisation $M = \partial g/\partial H|_{H=0}$ has a singular part in $-t$:

$$M \propto (-t)^{(d-y_H)/y_t}$$

i.e.

$$\beta = (d - y_H)/y_t. \tag{3.63}$$

Further discussion of the existence of spontaneous magnetisation is given by Niemeijer

and van Leeuwen (1976). The divergence of the susceptibility $\chi = \partial M/\partial H$ is given by $\chi \propto |t|^{-\gamma}$ ($H=0$): using (3.62(b)) again, one has

$$\gamma = (2y_H - d)/y_t. \tag{3.64}$$

The exponent δ, defined by $H \propto M^\delta$ at $t=0$, is similarly

$$\delta = y_H/(d - y_H). \tag{3.65}$$

Finally the specific heat exponent, α, can also be obtained from (3.62(b)):

$$\alpha = 2 - d/y_t. \tag{3.66}$$

The scaling laws

$$\alpha + 2\beta + \gamma = 2 \qquad \alpha + \beta(\delta+1) = 2 \qquad \gamma = \beta(\delta-1) \qquad \Delta = \beta\gamma \tag{3.67}$$

are therefore satisfied. Note also that, for a simple fixed point with the regularity properties assumed above, exponents are the same above and below T_c, because the value of λ_a is independent of the sign of Q_a (but see the discussion in §6.1.3).

Fixed points with more than two relevant variables describe multicritical points. The amplitude A in (3.61), called the crossover scaling function in this case, can describe the crossover towards a more stable critical fixed point. Examples are given in §6.

Finally, we remark that it is straightforward to obtain a form of the scaling equation of state (Widom 1965, Domb and Hunter 1965):

$$H = M^\delta h(t/M^{1/\beta}) \tag{3.68}$$

by taking $\partial/\partial H$ of (3.62) to obtain M. Note that we have dropped $\pm$ symbols here (as may also be done in equation (3.62(a)) because H is a regular function of t at fixed M (Lebowitz and Penrose (1968); see Griffiths (1972) for a review of analyticity properties and Brézin *et al* (1973b) for a discussion of regularity in the context of the RG).

3.2.2. Correlation functions. In the previous subsection, we considered the free energy and its derivatives, which are related to specific correlation functions as in, for example, equation (2.6). The treatment of arbitrary correlation functions using RG methods on a lattice is rather complicated. We give here only a brief review and refer the reader to Niemeijer and van Leeuwen (1976) for more details. The formalism in field theories is simpler (see §5). The main early reference to phenomenological scaling in correlation functions is Patashinskii and Pokrovskii (1966).

The most direct way to derive relations like (3.30) for correlation functions is to add 'source terms' equivalent to a spatially varying magnetic field, and consider the Hamiltonian

$$\mathscr{H} - \sum_m H_m s_m \tag{3.69}$$

leading to the free energy

$$G(\boldsymbol{K}, \{H_m\}) = -\ln \sum_{\{s\}} \exp\left(-\mathscr{H} + \sum H_m s_m\right). \tag{3.70}$$

Derivatives of this function generate all correlation functions of s:

$$G^{(p)}(m_1, \ldots, m_p; \boldsymbol{K}) = -\partial^p G/(\partial H_{m_1} \ldots \partial H_{m_p})\big|_{\{H\}=0}. \tag{3.71}$$

For example

$$G^{(2)}(m_1, m_2; \boldsymbol{K}) = \langle s_{m_1} s_{m_2} \rangle - \langle s_{m_1} \rangle \langle s_{m_2} \rangle \tag{3.72}$$

gives the two-spin correlation function.

Since we are interested in 'small' $\{H_m\}$, let us write

$$G(\boldsymbol{K}, \{H_m\}) = G + \Delta G \tag{3.73}$$

where G is just (3.23) and ΔG contains the dependence on $\{H_m\}$, so that $G^{(p)}$ is the derivative of ΔG. Following §3.2.1, we have $G' + \Delta G' = G + \Delta G$, i.e.

$$\Delta G' = \Delta G \tag{3.74}$$

where $\Delta G'$ contains:

(*a*) The additional spin-independent part of $\mathscr{H}'$, which we denote by $\Delta_0(\{H_m\})$ (which is extensive insofar as there are N H_m);

(*b*) The additional spin-dependent part of $\mathscr{H}'$. This will in general have a complicated form but let us assume to begin with that it looks like

$$-\sum H_\mu' \sigma_\mu$$

with

$$H_\mu' = H_\mu'(\{H_m\}, \boldsymbol{K}) \tag{3.75}$$

in analogy with (3.69). The formalism for the general case will be outlined later.

When differentiated with respect to H_μ', the (*b*) part of $\Delta G'$ produces $G^{(p)}(\mu_1, \ldots, \mu_p; \boldsymbol{K}')$. Differentiation of (3.74) with respect to H_m leads to relations for $G^{(p)}$ like (3.30) for g. In particular, let us consider the case $p=2$. (We now drop this superscript on $G^{(2)}$; no confusion with free energy G should arise.) Defining

$$C(m, n; \boldsymbol{K}) \equiv -\left.\frac{\partial^2 \Delta_0}{\partial H_m \partial H_n}\right|_{\{H\}=0} \tag{3.76}$$

and

$$T_{m^\mu} \equiv \left.\frac{\partial H_\mu'}{\partial H_m}\right|_{\{H\}=0} \qquad T_{mn^\mu} \equiv \left.\frac{\partial^2 H_\mu'}{\partial H_m \partial H_n}\right|_{\{H\}=0} \tag{3.77}$$

we have

$$C(m, n; \boldsymbol{K}) + \sum_{\mu, \nu} G(\mu, \nu; \boldsymbol{K}') T_{m^\mu} T_{n^\nu} + \sum_\mu G^{(1)}(\mu; \boldsymbol{K}') T_{mn^\mu} = G(m, n; \boldsymbol{K}). \tag{3.78}$$

To simplify the discussion further let us assume that $\mathscr{H}$ is even under $s_m \to -s_m$ (e.g. contains no external field). Then H_μ' is odd in H_m, and T_{mn^μ} vanishes. Equation (3.78) reduces to

$$G(m, n; \boldsymbol{K}) = C(m, n; \boldsymbol{K}) + \sum_{\mu\nu} G(\mu, \nu; \boldsymbol{K}') T_{m^\mu} T_{n^\nu}. \tag{3.79}$$

This recursion formula for G is rather similar to that for g. The analogy for a translation-invariant system can be made explicit by two steps.

(i) Rewrite (3.79) in terms of Fourier transforms. One obtains

$$\tilde{G}(\boldsymbol{k}; \boldsymbol{K}) = \tilde{C}(\boldsymbol{k}; \boldsymbol{K}) + b^{-d} \tilde{T}(\boldsymbol{k}; \boldsymbol{K}) \tilde{T}(-\boldsymbol{k}; \boldsymbol{K}) \tilde{G}(b\boldsymbol{k}; \boldsymbol{K}'). \tag{3.80}$$

The notation is as follows. Associate site $m(\mu)$ with a position vector $\boldsymbol{r}_m(\boldsymbol{\rho}_\mu)$ in its respective lattice structure with spacing $a(ba)$. Site μ can also be measured with respect to the lattice with spacing a, by $\boldsymbol{r}_\mu$ say. Clearly

$$\boldsymbol{r}_\mu - \boldsymbol{r}_\nu = b(\boldsymbol{\rho}_\mu - \boldsymbol{\rho}_\nu). \tag{3.81}$$

The Fourier components in (3.80) are defined for a translation-invariant system (G, etc, depending only on *differences* $\boldsymbol{r}_m - \boldsymbol{r}_n$) by

$$N\tilde{G}(\boldsymbol{k};\boldsymbol{K}) = \sum_{m,n} G(m,n;\boldsymbol{K}) \exp\left[\mathrm{i}\boldsymbol{k}.(\boldsymbol{r}_m - \boldsymbol{r}_n)\right] \tag{3.82(a)}$$

$$N\tilde{C}(\boldsymbol{k};\boldsymbol{K}) = \sum_{m,n} C(m,n;\boldsymbol{K}) \exp\left[\mathrm{i}\boldsymbol{k}.(\boldsymbol{r}_m - \boldsymbol{r}_n)\right] \tag{3.82(b)}$$

$$N'\tilde{G}(\boldsymbol{\varkappa};\boldsymbol{K}') = \sum_{\mu,\nu} G(\mu,\nu;\boldsymbol{K}') \exp\left[\mathrm{i}\boldsymbol{\varkappa}.(\boldsymbol{\rho}_m - \boldsymbol{\rho}_n)\right] \tag{3.82(c)}$$

and

$$\tilde{T}(\boldsymbol{k};\boldsymbol{K}) = \sum_{m} T_m{}^{\mu} \exp\left[\mathrm{i}\boldsymbol{k}.(\boldsymbol{r}_\mu - \boldsymbol{r}_m)\right]. \tag{3.82(d)}$$

Then (3.80) follows using (3.81) and $N'/N = b^{-d}$.

(ii) Rewrite the $\boldsymbol{K}$ dependence in terms of the fixed point and scaling variables. In the same spirit as the discussion at the end of §2.3.1, used extensively in the previous subsection, we assume that the functions C and T are regular in $\boldsymbol{K}$ close to the fixed point and in addition of short range, so that C and T are regular in $\boldsymbol{k}$.

First consider (3.80) at the fixed point $K_i' = K_i = K_i{}^*$, so that we can drop this argument from the equations. $\tilde{T}(\boldsymbol{k})$ can then be absorbed into a redefinition of $\tilde{G}$ and $\tilde{C}$ as follows. Note that $\tilde{T}(0)$ is just a magnetic eigenvalue, since it is just $\partial H'/\partial H\,|_{K=K^*}$:

$$\tilde{T}(0) = b^{y_H}. \tag{3.83}$$

Define a function $\psi(k)$ by

$$\psi(b\boldsymbol{k}) = b^{-y_H}\tilde{T}(\boldsymbol{k})\psi(\boldsymbol{k}) \qquad \psi(0) = 1. \tag{3.84}$$

Given $\tilde{T}$ which is assumed to be Taylor-expandable:

$$b^{-y_H}\tilde{T}(k) = 1 + \sum \tau_{i_1 \ldots i_p} k_{i_1} \ldots k_{i_p} \tag{3.85}$$

one can write an explicit solution for ψ:

$$\psi(\boldsymbol{k}) = 1 + \sum a_{i_1 \ldots i_p} k_{i_1} \ldots k_{i_p} \tag{3.86(a)}$$

with

$$a_i = \frac{\tau_i}{b-1} \tag{3.86(b)}$$

and

$$a_{i_1 \ldots i_p} = (\tau_{i_1 \ldots i_p} + \sum_{n=1}^{p-1} a_{i_1 \ldots i_{n-p}} \tau_{i_1 \ldots i_n})\,(b^p - 1)^{-1}. \tag{3.86(c)}$$

Defining finally

$$\mathscr{G}(\boldsymbol{k}) = \psi(\boldsymbol{k})\psi(-\boldsymbol{k})\tilde{G}(\boldsymbol{k}) \tag{3.87(a)}$$

$$\mathscr{C}(\boldsymbol{k}) = \psi(\boldsymbol{k})\psi(-\boldsymbol{k})\tilde{C}(\boldsymbol{k}) \tag{3.87(b)}$$

we see that (3.80) can be written

$$\mathscr{G}(\boldsymbol{k}) = \mathscr{C}(\boldsymbol{k}) + b^{-d+2y_H}\mathscr{G}(b\boldsymbol{k}). \tag{3.88}$$

Following precisely the analysis for the free energy g, we infer that the singular part of $\mathscr{G}(\boldsymbol{k})$ (around $\boldsymbol{k} = 0$) has the behaviour

$$\mathscr{G}_s(\boldsymbol{k}) = \mathscr{A}\,|\,\boldsymbol{k}\,|^{\,d-2y_H} \tag{3.89}$$

where $\mathscr{A}$ is a critical amplitude similar to A for g (cf (3.44) and (3.45)). Since

$\mathscr{G}$ and G are related by (3.87(a)) and ψ is regular at $\boldsymbol{k}=0$, we obtain a singularity in G:

$$\tilde{G}_s(\boldsymbol{k}) = \mathscr{A}\,|\boldsymbol{k}|^{d-2y_H}. \tag{3.90}$$

Fourier transforming back to $G(m, n)$, this implies that as $|\boldsymbol{r}_m - \boldsymbol{r}_n| \to \infty$:

$$G(m, n) \sim |\boldsymbol{r}_m - \boldsymbol{r}_n|^{-2(d-y_H)}. \tag{3.91}$$

Using the definition of η in

$$G(m, n) \sim |\boldsymbol{r}_m - \boldsymbol{r}_n|^{-(d-2+\eta)} \tag{3.92}$$

we infer

$$\eta = d + 2 - 2y_H. \tag{3.93}$$

This result and (3.65) imply the hyperscaling law of Fisher (1967a, 1969):

$$\delta = (d+2-\eta)\,(d-2+\eta)^{-1}. \tag{3.94}$$

For a system not at criticality we have at least one relevant temperature-like scaling field, which we again call $t(\sim T - T_c)$. Following the analysis for $g(Q_1, Q_2)$, the amplitude will now be a function of $t\,|\boldsymbol{k}|^{-y_t}$:

$$\mathscr{A} = \mathscr{A}(t\,|\boldsymbol{k}|^{-y_t}) \tag{3.95}$$

so that

$$\mathscr{G}(\boldsymbol{k}, t) = |\boldsymbol{k}|^{-2+\eta}\mathscr{A}(t\,|\boldsymbol{k}|^{-y_t}). \tag{3.96}$$

To extract the exponent ν from this equation, we need a definition of the correlation length ξ. It may be defined, for example, as an average correlation range:

$$(\xi/\text{lattice spacing})^2 = \sum_m m^2 G(m, 0) / \sum_m G(m, 0)$$

or by the position of a singularity of $\mathscr{A}^{\pm}$ (in the complex plane):

$$\mathscr{A}^{\pm}(\mathrm{i}t\,|\,\xi\,|^{y_t}) = \infty.$$

In both cases,

$$\xi \propto t^{-1/y_t} \tag{3.97}$$

so that

$$\nu = y_t^{-1} \tag{3.98}$$

a result arrived at earlier (equation (3.12)).

Thus we see that, after Fourier transform, the analysis of scaling behaviour for correlation functions can be done by analogy with scaling of the free energy. Scaling laws and formulae for scaling functions, such as $\mathscr{A}(t\,|\boldsymbol{k}|^{-y_t})$, can be derived. Further remarks are in order.

(i) In obtaining (3.96), apart from assumptions of regularity, etc, we made one assumption which is certainly not true, i.e. that the new interaction induced by $\Sigma_m H_m s_m$ is of the form $\Sigma_\mu H_\mu' \sigma_\mu$; in general, three-spin and higher interactions are also present in $\mathscr{H}'$. This complication is handled by extending the notion of scaling fields Q_a to wavevector-dependent fields $Q_a(\boldsymbol{k})$, transforming as

$$Q_a'(b\boldsymbol{k}) = b^{y_a} Q_a(\boldsymbol{k}). \tag{3.99}$$

These are obtained by perturbing away from the fixed point with space-varying interactions $S_{i,\,m}$, of cluster type i located by site m, so that

$$\delta K_{i,\,\mu}' = M_{i,\,\mu}^{\;j,\,m}(K^*)\,\delta K_{j,\,m}. \tag{3.100}$$

Given a translation-invariant fixed-point Hamiltonian (K^*) we can diagonalise M in (3.100) after Fourier transforming. The (k-dependent) eigenvectors can then be extended to non-linear scaling fields $Q_a(\boldsymbol{k})$. The two-spin correlation function $\partial^2 G/\partial H_m \partial H_n$ can then be written as a sum over correlation functions involving scaling operators, i.e. as a sum over $\partial^2 G/\partial Q_a(\boldsymbol{k})\partial Q_b(\boldsymbol{k})$. Each of these terms will scale as in (3.89) with the appropriate power $|\boldsymbol{k}|^{d-y_a-y_b}$. The dominant term for low $|\boldsymbol{k}|$ will be that for largest y_a+y_b, i.e. for the dominant magnetic exponent y_H, and the previous result (3.90) remains. Details are given in Niemeijer and van Leeuwen (1976).

(ii) Similar analysis can be given for correlation functions involving many spins, and products of, for example, nearest-neighbour spins, corresponding to energy density correlations.

(iii) The discussion of correlation functions in field theories can be much simpler than in lattice models. It *is* possible to have the field variable $\phi(x)$ (analogous to s_m) as a scaling interaction, so that RG equations for different correlation functions of $\phi(x)$ are decoupled (as we have assumed initially to obtain equations (3.91) and (3.96)). Formal scaling properties in the presence of a magnetic field and below T_c are also simpler to discuss in field theory (see §5 and the references in §6.1.4).

(iv) Let us consider the regularity properties of the two-spin correlation function:

$$\mathscr{G}(\boldsymbol{k}, t) = |\boldsymbol{k}|^{-2+\eta}\mathscr{A}_k^{\pm}(t|\boldsymbol{k}|^{-y_t}) \tag{3.101(a)}$$

$$= t^{-\gamma}\mathscr{A}_t^{\pm}(|\boldsymbol{k}||t|^{-1/y_t}) \tag{3.101(b)}$$

where

$$\gamma = (2-\eta)/y_t = \nu(2-\eta). \tag{3.102}$$

$\mathscr{G}$ is given in terms of the Fourier transform of $G(m, n)$, up to regular terms in $\boldsymbol{k}^2$. Away from a critical point $G(m, n)$ has finite range and so in the Fourier transform, $\exp[\mathrm{i}\boldsymbol{k}.(\boldsymbol{r}_m-\boldsymbol{r}_n)]$ may be expanded as a power series in $\boldsymbol{k}(\boldsymbol{r}_m-\boldsymbol{r}_n)$, the sum on $(m-n)$ for each term being convergent, i.e. $\mathscr{G}(\boldsymbol{k})$ is regular in $|\boldsymbol{k}|$ away from a critical point. Hence $\mathscr{A}_t^{\pm}$ in (3.101(b)) is regular in its argument.

Let us consider however the amplitude $\mathscr{A}_k$ in (3.101(a)). So far in this review the variable with respect to which the amplitude is defined has corresponded to a perturbation away from the critical point, e.g. the variables H and t in (3.63(a) and (b)). Given that the system is *already* away from its critical point, we expect to be able to make *regular* perturbations in *other* variables according to the qualitative arguments in §3.1 and the explicit calculation in §3.2.1. However, in (3.101(a)), $\mathscr{G}(\boldsymbol{k}, 0)$ represents a system *at* its critical point and there is no basis whatsoever for assuming that the perturbation in t will be regular, i.e. $\mathscr{A}_k$ in equation (3.101(a)) is not in general a regular function.

The operator product expansion (Kadanoff 1969, Wilson 1969) enables one to obtain the low t behaviour of $\mathscr{A}_k$. The basic idea is that for $t|\boldsymbol{k}|^{-y_t}$ small, i.e. $|\boldsymbol{r}_m-\boldsymbol{r}_n| \ll \xi$, the product of operators of the type $O_m{}^a$, the Fourier transform of the interaction conjugate to the scaling field $Q_a(\boldsymbol{k})$, can be expressed as a sum over such operators times coefficients which are functions of $(\boldsymbol{r}_m-\boldsymbol{r}_n)$:

$$O_m{}^a O_n{}^b = \sum_c C_c{}^{ab}(\boldsymbol{r}_m-\boldsymbol{r}_n) O_m{}^c. \tag{3.103}$$

The critical exponents of the scaling fields then determine the power of $(\boldsymbol{r}_m-\boldsymbol{r}_n)$ in $C_c{}^{ab}(\boldsymbol{r}_m-\boldsymbol{r}_n)$. This approach has been strongly developed in field theory: Zimmer-

mann (1971, 1973a, b) and Brézin *et al* (1976a) contain further references; see also Kadanoff (1976a). A typical result is

$$\mathscr{G}(\boldsymbol{k}, t) = |\boldsymbol{k}|^{-2+\eta}[A^{\pm} + B^{\pm} t |\boldsymbol{k}|^{-1/\nu} + C^{\pm} |t|^{1-\alpha} |\boldsymbol{k}|^{-(1-\alpha)/\nu} + \ldots] \qquad |t| \ll |\boldsymbol{k}|^{1/\nu} \qquad (3.104)$$

where α, the specific heat critical exponent as in equation (3.66), arises because the operator containing the energy density appears in the right-hand side of (3.103), behaving like $\partial g/\partial t \sim t^{1-\alpha}$.

(v) Universality properties for correlation functions can be discussed by the same method used for the free energy.

Finally, we emphasise again that although we have talked in terms of free energy and correlation functions near a critical point, one may envisage calculations of analogous physical quantities using the RG in other many-body problems. In general the physics need not be dominated by fixed points and scaling, of course.

4. Real-space methods in lattice models

4.1. Introductory remarks

4.1.1. Lattice and field models. The preceding sections deal with ideas and general formalisms. We now present some details on some of the methods for calculating the transformation functions $f_i(\boldsymbol{K})$.

There are basically two approaches, complementary in a number of ways. The lattice methods, the simplest example of which was presented in §2.3.2, have the following advantages: (i) conceptual and technical simplicity, (ii) validity in any integer dimension d, in particular $d = 2$ so that comparison with exact results is possible, and (iii) ability to calculate non-universal quantities of interest, such as the critical temperature T_c, and behaviour over the entire temperature range without concern about the 'size' of the critical region.

In the field theoretic approach one works with a field, i.e. a continuous spin density, rather than the discrete spin variables of, for example, the Ising model. The advantages are (i) existence of powerful well-developed tools in field theory, (ii) easy identification of d_c, the critical dimension below which mean field approximations fail, (iii) ability to continue analytically in dimension to obtain ϵ expansions in $d_c - \epsilon$ dimensions, (iv) simplicity of the fixed-point Hamiltonians, and the analysis of perturbations about these fixed points and crossover phenomena, (v) analytic structure in other parameters of the theory, e.g. n, the number of components of the field (spin) and, more recently, (vi) calculation of critical behaviour and T_c, for Heisenberg-like models, near the critical dimension above which $T_c > 0$.

Due to the simplicity of the lattice methods, a self-contained presentation of the general formalism and some successful examples can be accommodated in some detail. By contrast, it is beyond the scope of this review to give such an account of field theoretic methods. Instead, we attempt only to give the newcomer some notion of the vocabulary and the approach. Remarks on the relationships between lattice and field models are contained in the discussion on field theories in §5.

We deal only with Ising-like models here; applications to other models can be found in §6.

4.1.2. General formulation. The simplest example of a renormalisation group transformation was given in §2.3.2: decimation. There is, however, a major flaw to this method, which will be discussed later. To incorporate more generally the central idea of reducing the 'density' of degrees of freedom one introduces a set of new spins σ with larger lattice spacing than s and defines a new effective Hamiltonian $\mathscr{H}'(\sigma)$ by

$$\exp(-\mathscr{H}'(\sigma)) = \sum_{\{s\}} W(\sigma, s) \exp(-\mathscr{H}(s)) \tag{4.1}$$

where $W(\sigma, s)$ is any function of σ and s such that

(i) $\mathscr{H}'(\sigma)$ remains real, a sufficient but not necessary condition for which is $W(\sigma, s) \geqslant 0$ for all σ and s.

(ii)

$$\sum_{\{\sigma\}} W(\sigma, s) = 1 \tag{4.2}$$

so that the partition function for the new system is the same as before:

$$Z = \sum_{\{\sigma\}} \exp(-\mathscr{H}') = \sum_{\{s\}} \exp(-\mathscr{H}). \tag{4.3}$$

(iii) General, qualitative features (e.g. symmetries) of physics are retained.

The last point is rather vague but extremely important. Apart from obviously bad choices, e.g. ones that do not respect translational invariance or rotational symmetry of the original system, we would like to emphasise the point by illustrating other pitfalls. In principle, there is nothing to stop one from choosing σ to take on, say, *three* values $(\pm 1, 0)$. But such a system may display qualitative features drastically different from the spin $\frac{1}{2}(s = \pm 1)$ Ising model, e.g. tricritical phenomena (see §6). The consequence of such a choice is that one might be led to behaviour totally unrelated to the original system. Another example involving ferromagnetic and antiferromagnetic fixed points was discussed in §2.3.2. Further examples are described in van Leeuwen (1975) and Burkhardt and Knops (1977).

There are many more subtle difficulties. For example, in straightforward decimation, either $\eta = d - 2$ or no fixed-point Hamiltonian ($\mathscr{H}^*$) exists. This conclusion comes from considering the pair-correlation function $G^{(2)}$ at the critical point (equation (3.72) with $\langle s \rangle = 0$):

$$G^{(2)} \equiv \langle s_n s_m \rangle \equiv \left[\sum_{\{s\}} s_n s_m \exp(-\mathscr{H}(s)) \right] / Z. \tag{4.4}$$

Choose s_n and s_m to be survivors of the decimation so that, in the new system, they are just σ_ν, σ_μ. With this choice, in (4.4) the configuration sum over the decimated spins, $\{\tilde{\sigma}\}$ can be done:

$$\sum_{\{s\}} s_n s_m \exp(-\mathscr{H}(s)) = \sum_{\{\sigma\}} \sigma_\nu \sigma_\mu \left[\sum_{\{\tilde{\sigma}\}} \exp(-\mathscr{H}) \right] = \sum_{\{\sigma\}} \sigma_\nu \sigma_\mu \exp(-\mathscr{H}'(\sigma)). \tag{4.5}$$

Since Z remains the same, we have

$$\langle s_n s_m \rangle = \langle \sigma_\nu \sigma_\mu \rangle. \tag{4.6}$$

At the critical temperature, for large separations, these behave like $A|n-m|^{2-d-\eta}$ and $A'|\nu-\mu|^{2-d-\eta}$ respectively. Suppose an $\mathscr{H}^*$ exists, then $\mathscr{H} = \mathscr{H}' = \mathscr{H}^*$ leads to $A = A' = A^*$. On the other hand

$$|n-m| = b|\nu-\mu| \tag{4.7}$$

where b is the ratio of the new lattice spacing to the old. Putting these equations together, we obtain either $G^{(2)*} \propto A^* = 0$, which is absurd, or

$$b^{2-d-\eta} = 1. \tag{4.8}$$

Since η is not, in general, $2-d$, we conclude that if a decimation transformation were performed exactly, no non-trivial $\mathscr{H}^*$ will be reached.

Another difficulty is associated with the possible existence of redundant interactions. These may arise because we permit $\mathscr{H}(s)$ to be in principle an arbitrary function of s and parametrise it by a set of coupling constants K_i as in equation (3.1). In general, however, this appears to be an overcomplete specification in the sense that for any given fixed point K_i^* there may exist a whole manifold of physically equivalent fixed points. This point was first appreciated in field theory (Wilson and Kogut 1974 (last paragraph in the appendix), Wegner 1974a, 1976, Zia and Wallace 1975a). An example in decimation in a lattice model is given in §4.2.1 (Bell and Wilson 1974, Wilson 1975, T L Bell unpublished; see also Subbarao 1976a). After approximations are made in lattice calculations, the redundancy may not persist, but undesirable effects such as spurious relevant interactions may possibly remain (see also Green 1977).

Apart from these general problems, in practice, one must choose $W(\sigma, s)$ in such a way that the configuration sum over $\{s\}$ can be done in some approximation which produces an $\mathscr{H}^*$ about which linearisation may be done. Typically, a factorisable form is chosen:

$$W(\sigma, s) = \prod_{\nu,\, n \in \mathscr{N}} w(\sigma_\nu, s_n) \tag{4.9}$$

where $\mathscr{N}$ is a set of b^d sites 'surrounding' the new site ν. Note that (4.9) incorporates translational invariance. A successful scheme proceeds from here by deriving 'good approximations' to the recursion relations. In the following, we present three of a number of such schemes.

4.2. Specific schemes

4.2.1. Kadanoff–Wilson decimation. To circumvent the difficulty of $\eta = 2-d$ in simple decimation, L P Kadanoff (unpublished), Kadanoff and Houghton (1975), Bell and Wilson (1975) and Wilson (1975) generalise the procedure by choosing

$$w(\sigma_\nu, s_n) = \tfrac{1}{2}(1 + p\sigma_\nu s_{n(\nu)}) \tag{4.10}$$

where p is an arbitrary parameter to be determined and $n(\nu)$ labels the surviving site in the simple decimation scheme (which corresponds to the choice $p=1$). Before looking at some results of this approach, let us make several general remarks about this choice.

A general feature of decimation, like schemes such as (4.10), is that for the fixed-point Hamiltonian new correlation functions are proportional to the old:

$$\begin{aligned} Z\langle \sigma_{\mu_1} \ldots \sigma_{\mu_K} \rangle &= \sum_{\{\sigma\}} \sigma_{\mu_1} \ldots \sigma_{\mu_K} \exp\left(-\mathscr{H}^*(\sigma)\right) \\ &= \sum_{\{\sigma,\, s\}} \sigma_{\mu_1} \ldots \sigma_{\mu_K} \prod \tfrac{1}{2}(1 + p\sigma s) \exp\left(-\mathscr{H}^*(s)\right) \\ &= p^k \sum_{\{s\}} s_{n(\mu_1)} \ldots s_{n(\mu_k)} \exp\left(-\mathscr{H}^*(s)\right) \\ &= p^k Z\langle s_{n(\mu_1)} \ldots s_{n(\mu_k)} \rangle. \end{aligned}$$

RG transformations with this property are called *linear* transformations. The proportionality factor p^k now saves us from the $2-d$ disaster. If we run through the arguments concerning the pair-correlation function again, we find (4.8) modified:

$$p^2 b^{2-d-\eta} = 1 \tag{4.11}$$

so that a non-trivial $G^{(2)}$ exists only if we choose a *special* value of p, p^* say, obeying

$$p^* = b^{(\eta+d-2)/2}. \tag{4.12}$$

(In most cases $p^* > 1$ so that $W(\sigma, s) \geqslant 0$ (see §4.1.2) is no longer automatic.)

Analysing further, we find that if an exact $\mathscr{H}^*$ exists, then there is a one-parameter family of them, $\mathscr{H}_\lambda{}^*$ (Wilson 1975):

$$\exp(-\mathscr{H}_\lambda{}^*(t)) \equiv \sum_{\{s\}} \prod_n \tfrac{1}{2}(1+\lambda t_n s_n) \exp(-\mathscr{H}^*(s)) \tag{4.13}$$

where the product, unlike decimation, runs over *all sites*. We have therefore an example of redundant interactions. To show that $\mathscr{H}_\lambda{}^*$ is also a fixed point of the generalised decimation we employ the identity

$$\sum_{\{\sigma\}} \prod_\nu \tfrac{1}{2}(1+\lambda\tau_\nu\sigma_\nu) \prod_\mu \tfrac{1}{2}(1+p^*\sigma_\mu s_{n(\mu)}) = \sum_{\{t\}} \prod_\nu \tfrac{1}{2}(1+p^*\tau_\nu t_{n(\nu)}) \prod_m \tfrac{1}{2}(1+\lambda t_m s_m) \tag{4.14}$$

so that

$$\begin{aligned}
\exp(-\mathscr{H}_\lambda{}^*(\tau)) &= \sum_{\{\sigma\}} \prod \tfrac{1}{2}(1+\lambda\tau\sigma) \exp(-\mathscr{H}^*(\sigma)) \\
&= \sum_{\{\sigma, s\}} \prod \tfrac{1}{4}(1+\lambda\tau\sigma)(1+p^*\sigma s) \exp(-\mathscr{H}^*(s)) \\
&= \sum_{\{t, s\}} \prod \tfrac{1}{4}(1+p^*\tau t)(1+\lambda t s) \exp(-\mathscr{H}^*(s)) \\
&= \sum_{\{t\}} \prod \tfrac{1}{2}(1+p^*\tau t) \exp(-\mathscr{H}_\lambda{}^*(t)).
\end{aligned} \tag{4.15}$$

The consequence of such a fixed line is that, in an exact calculation, p^* should come out to be independent of the $\mathscr{H}^*$ parametrised on the line by, say, the nearest-neighbour coupling K^*. In practice, approximations to the recursion relations induce a non-trivial dependence $p^*(K^*)$.

We now present the typical approximation schemes. It is obvious that the $T = \infty$ (or non-interacting) Hamiltonian is a fixed point: $\mathscr{H}^* = 0$. Thus it is reasonable to try an expansion about this point ($K = 0$), i.e. a high-temperature expansion. For definiteness, let us concentrate on a square lattice and consider a decimation whereby one site out of every $b \times b$ block survives. Carrying out the transformation specified by (4.11), we write down the first few new interactions to the lowest non-trivial order:

$$\begin{aligned}
K' &= p^2 K^b (1 + O(K^2)) \\
L' &= p^2 \binom{2b}{b} K^{2b} (1 + O(K^2)) \\
M' &= p^2 K^{2b} (1 + O(K^2)) \\
N' &= p^2 \binom{3b}{b} K^{3b} (1 + O(K^2))
\end{aligned} \tag{4.16}$$

where $L'(M', N')$ is the new next (next–next, next–next–next) nearest-neighbour

coupling and the binomial coefficients count the number of shortest paths between the relevant new sites. Being a systematic expansion, it can be used as a basis of the approximation scheme where terms up to a certain order are kept. To find the fixed point, L, M, N, etc, type terms are needed on the right-hand side of (4.16) for consistency. For example, if we wish to keep K^4 terms with $b=2$, the necessary equations are

$$K'=p^2(K^2+(16/3)K^4+M) \qquad L'=6p^2K^4 \qquad M'=p^2K^4. \tag{4.17}$$

The serious pursuer of such a programme should use the variables $\tanh K$, etc, where combinatorics simplify considerably.

We summarise now Wilson's (1975) work. He chose to decimate every other spin on the two-dimensional square lattice, leading to $b=\sqrt{2}$. Keeping only the lowest orders, K^2 and L (the next-nearest-neighbour coupling), the recursion relations

$$K'=p^2(2K^2+L) \tag{4.18}$$

$$L'=p^2K^2 \tag{4.19}$$

lead to

$$K^*=(2p^{*2}+p^{*4})^{-1}. \tag{4.20}$$

This relatively strong dependence $p^*(K^*)$ is the result of the approximation. Improvements are possible by including a larger number of interactions involving clusters of spins. Going to fairly high order Wilson succeeded in finding a p^* which varies little with K^*, in contrast to (4.20). These calculations are quite tedious, though straightforward in principle, so that computers are used. Referring the interested reader to the original paper for any details, we simply quote the excellent results:

$$\eta=0{\cdot}2497 \qquad \nu=0{\cdot}998 \tag{4.21}$$

for a range of K^* around 0·3. Note that these K^* are not to be confused with the exact $K_c\simeq 0{\cdot}4409$. The recursion relations take the latter point ($K=K_c$, $0=L=M=\ldots$) to some point on the fixed line $\mathscr{H}_\lambda{}^*$.

A less ambitious scheme is to fix p^* by the exact $\eta=\frac{1}{4}$. Then one can ask what are K^* and ν in some approximation. For example, if we wish to go beyond lowest order without introducing an L interaction, we can use a 3×3 block. Writing

$$\omega=\tanh K, \qquad \text{etc} \tag{4.22}$$

the recursion relation up to ω^5 is

$$\omega'=3^{1/4}\omega^3(1+12\omega^2) \tag{4.23}$$

from which we get

$$\omega^*\sim 0{\cdot}46.$$

Within our approximation, this lies on the K axis, so that we *can* compare it with

$$\omega_{\text{exact}}\simeq 0{\cdot}414. \tag{4.24}$$

This is in reasonable agreement, considering how crude the approximation is. The exponent ν is poor: 0·74 (cf $\nu_{\text{exact}}=1$). Not to be taken completely seriously, this simple example nevertheless illustrates the basic ideas behind the programme.

4.2.2. Niemeijer–van Leeuwen cells. A particularly appealing choice of the weight factor was introduced by Niemeijer and van Leeuwen (1973, 1974): the sign rule. Divide the lattice into cells (labelled by μ) of b^d sites and define each new cell spin by

$$\sigma_\mu = \operatorname{sgn}\left(\sum_{n\in\mathcal{N}(\mu)} s_n\right) \tag{4.25}$$

where $\mathcal{N}(\mu)$ is the set of sites in cell μ. Note that, unless b^d is odd, some convention must be made to make (4.25) well-defined. We also mention in passing that for a d-dimensional hypercubic lattice b^d must be a sum of d squares:

$$b^d = \sum_{i=1}^{d} n_i^2 \qquad n_i \text{ integer}$$

(e.g. $b=\sqrt{5}$ in $d=2$) if the lattice structure is to be retained at the cell level.

The choice (4.25) is most natural for the two-dimensional Ising model on the triangular lattice where a cell can be just the three-site cluster. In this case, every site can be labelled by (i, μ) with $i=1, 2, 3$. (We suppress the cell label whenever it is clear.) The weight factor is, for each cell

$$w(\sigma, s) = \tfrac{1}{2}[1 + \tfrac{1}{2}\sigma_\mu(s_1+s_2+s_3-s_1s_2s_3)]. \tag{4.26}$$

Due to the presence of the $s_1s_2s_3$ term, correlation functions of σ and s are no longer linearly related. This is an example of a non-linear transformation: troubles with $\eta=2-d$ do not arise.

To proceed efficiently, the configuration sum $\{s\}$ of any function of s, $A(s)$, with this weight factor may be written as

$$\sum_{\{s\}} W(\sigma, s)A(s) = \sum_{\{\bar{s}\}} A(s(\sigma, \bar{s})) \tag{4.27}$$

where $\bar{s}$ takes on four values per cell:

$$\begin{aligned} \bar{s}&=1 \qquad \sigma = -s_1 = s_2 = s_3 \\ \bar{s}&=2 \qquad \sigma = s_1 = -s_2 = s_3 \\ \bar{s}&=3 \qquad \sigma = s_1 = s_2 = -s_3 \\ \bar{s}&=4 \qquad \sigma = s_1 = s_2 = s_4. \end{aligned} \tag{4.28}$$

With these definitions, $s_i(\sigma, \bar{s}) = (1-2\delta_{\bar{s}i})\sigma$. (Note that, together with the two values of σ, the eight configurations of s per cell are retrieved.)

As with any scheme, approximations are necessary for arriving at simple enough recursion relations; the configuration sum in (4.1) is no more 'do-able' than the one without $W(\sigma, s)$. Here, there are three main types of approximations: (*a*) finite lattice, (*b*) cumulant expansion, and (*c*) cluster approximation.

(*a*) The simplest procedure is to take a finite lattice (of cells). For example, Tjon (1974) took the symmetric four-cell lattice in figure 7. Nearest neighbour, second neighbour and four spin couplings can be accommodated at the cell level. The RG transformation is defined by starting with these types of coupling at the *site* level and working out the required effective *cell* couplings by exact (or computer) analysis. Fixed points, critical exponents, etc, are then found by the usual methods of §3.

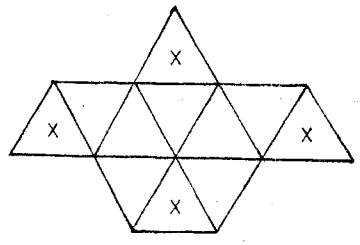

Figure 7. Finite lattice used by Tjon (1974); 'X-ed' triangles are cells.

It should be emphasised that this RG transformation on a finite lattice is distinct from obtaining thermodynamic quantities directly for a finite lattice. This is seen in the comparisons (figure 8) of the specific heat for the exact result, this simple approximation and the 12-spin finite lattice. Improvements occur (Subbarao 1975) if larger lattices are used, but convergence appears to be slow.

A variation of the method is to impose periodic boundary conditions. It is hoped that such conditions would reduce the spurious boundary effects and better approximate the infinite lattice. When this idea was applied to the four-cell lattice, no non-trivial fixed point could be found (J A Tjon 1976, private communication (see Niemeijer and van Leeuwen 1976)). The difficulty probably lies in that the lattice is too small for the inclusion of important interactions, i.e. L does not exist since the periodic boundary conditions would make cells 1 and 3 nearest neighbours.

Making a convention for the sign rule on a four-site cell, Nauenberg and Nienhuis (1974a, b), and Nienhuis and Nauenberg (1975) applied this method (with periodic boundary conditions) to a four-cell *square lattice* (16 sites). Now, L does exist. Critical exponents, the critical surface, free energy, specific heat and equation of state agree with exact results and series expansions to within a few per cent.

(*b*) The cumulant expansion attempts to deal with the infinite lattice. First, the Hamiltonian is separated into intra-cell interactions ($\mathscr{H}_0$) and inter-cell ones (V):

$$\mathscr{H} = \mathscr{H}_0 + V.$$

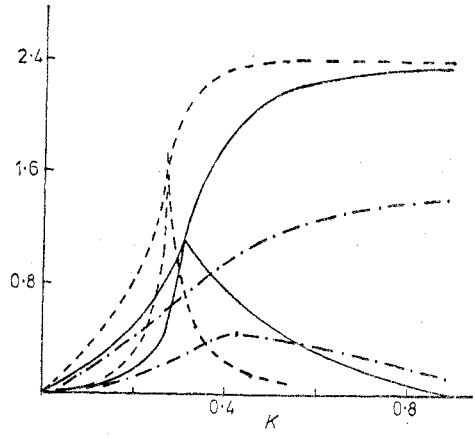

Figure 8. Internal energy and specific heat for the pure Ising system. Broken curve, exact solution; full curve, renormalisation results of Tjon (1974); chain curve, results for 12-spin lattice.

To be specific, let us consider a system with only nearest-neighbour interactions in an external field H:

$$\mathscr{H}_0(s) = \sum_{\mu} \hbar_0(s, \mu) = -\sum_{\mu} (K \sum_{\langle i,j \rangle} s_{i,\mu} s_{j,\mu} + H \sum_{i} s_{i,\mu}). \tag{4.29}$$

Since $\exp(-\mathscr{H}_0)$ factorises into products of $\exp(-\hbar_0)$ over cells, the configuration sum over $\{\bar{s}\}$ can now be done explicitly:

$$\sum_{\{\bar{s}\}} W(\sigma, s) \exp(-\mathscr{H}_0) = Z_0(\sigma) \tag{4.30}$$

where

$$Z_0(\sigma) = \prod_{\mu} Z_0(\sigma_\mu) = \prod_{\mu} [\exp(3K + 3H\sigma_\mu) + 3 \exp(-K + H\sigma_\mu)]. \tag{4.31}$$

Next, write

$$\sum_{\{\bar{s}\}} W(\sigma, s) \exp(-\mathscr{H}) = Z_0(\sigma) \langle \exp(-V) \rangle_0 \tag{4.32}$$

where we define the $\langle\ \rangle_0$ average of an operation O by

$$\langle O \rangle_0 \equiv \Big[\sum_{\{\bar{s}\}} W(\sigma, s)\, O \exp(-\mathscr{H}_0) \Big] / Z_0(\sigma). \tag{4.33}$$

The average of an exponential is then expressed in the cumulant expansion:

$$\langle \exp(-V) \rangle_0 = \exp(-\langle V \rangle_0 + \tfrac{1}{2}\langle V^2 \rangle_0 - \tfrac{1}{2}\langle V \rangle_0^2 + \ldots). \tag{4.34}$$

A systematic approximation scheme now emerges.

As an example, let us consider the first approximation $\langle V \rangle_0$ when the external field $H = 0$. Since V consists of a sum of the inter-cell nearest-neighbour site bonds, it is clear that there will be only nearest-neighbour cell bonds. Referring to figure 9

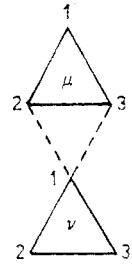

Figure 9. Inter-cell bonds 1–2 and 1–3 between cells μ and ν.

we see that two terms in V contribute to the interaction between the cells μ and ν. In this case, all of the configuration sum $\{\bar{s}\}$ (equation (4.27)) cancels except

$$Z_0^{-1}(\sigma_\mu) Z_0^{-1}(\sigma_\nu) \sum_{\{\bar{s}_\mu, \bar{s}_\nu\}} K(s_{2,\mu} + s_{3,\mu}) s_{1,\nu} \exp(\hbar_0(s, \mu) + \hbar_0(s, \nu)) \tag{4.35}$$

which leads to

$$K' = 2K \left(\frac{1+R}{1+3R} \right)^2 \qquad R \equiv \exp(-4K). \tag{4.36}$$

At this order, no other interactions play a role so that the non-trivial K^* of the

transformation (4.36) may be compared with K_c of the exact result (Syozi 1972, p280):

$$K^* \simeq 0{\cdot}336 \qquad \text{cf} \quad K_c = (\ln 3)/4 \simeq 0{\cdot}274. \tag{4.37}$$

The comparison of the exponent ν is more favourable:

$$\nu \simeq 0{\cdot}89 \tag{4.38}$$

especially considering that this is just the first non-trivial approximation. It is easy to include the external field H and find δ. Unfortunately, δ is extremely sensitive to small variations of the magnetic eigenvalue λ_H from the exact value of $3^{15/16} \simeq 2{\cdot}8$. Instead of comparing δ, we quote the result $\lambda_H \simeq 3{\cdot}04$.

Of course, one can go on to the higher orders. Other interactions must be introduced and, though straightforward, the calculation becomes quite laborious. Niemeijer and van Leeuwen showed that there are improvements at the second order. Hsu *et al* (1975) found, however, that the third-order results are worse than the second using a 3×3 block on a square lattice (see also Sudbø and Hemmer 1976).

Another remarkable result was found by Kadanoff and Houghton (1975) using the cumulant expansion with a parameter (p)-dependent weight factor. Working with a 2×2 block on a square lattice, they modified the weight (cf (4.26)) in the spirit of (4.11):

$$w(\sigma, s) = \tfrac{1}{2}\{1 + \sigma[(p_1 + p_3 s_1 s_2 s_3 s_4)(s_1 + s_2 + s_3 + s_4)]\} \tag{4.39}$$

where p_1 and p_3 are related to the parameter p by

$$\tanh\left(p \sum_i s_i\right) = \left(p_1 + p_3 \prod_i s_i\right) \sum_i s_i. \tag{4.40}$$

The motivation for this choice comes from the (un-normalised) weight of exp $(p\sigma \Sigma_i s_i)$.

If one could solve the exact recursion relations, the resultant exponents should be independent of p. Instead, at the second-order level of the cumulant expansion, there is a dependence of, say, η on p. The choice of p is then fixed by another dependence of η on p—a generalised version of (4.12). The result is a remarkable agreement to 0·4%. When this value of p is used to determine ν, the agreement is a phenomenal 4 parts in 10^5!

(*c*) Although both are successful, it is difficult to understand the basis on which either of the above schemes work so well. The finite lattice methods are not systematic. The cumulant expansion, apart from questions about convergence, should be good only for small K, yet K^* is not small. Thus the cluster approximation is developed to avoid being restricted to finite lattices and to exploit the fact that only K^* seems to be of the order of unity.

Though not difficult conceptually, the combinatoric technical details are beyond the scope of this review. We attempt to present the gist of the approximation and the best results obtained so far.

For definiteness, let us concentrate on the nearest-neighbour interaction K'; the general method applies to any interaction in fact. If we calculate a particular K' using a finite cluster $\boldsymbol{a}$ of cells (which includes the particular pair in question), then it depends on $\boldsymbol{a}$: $K'(\boldsymbol{a})$. But it is hoped that it will converge to K' when $\boldsymbol{a}$ is taken to the infinite lattice. To express this property, we write the Ursell expansion

$$K' = \sum_{\boldsymbol{c}} U(\boldsymbol{c}) \tag{4.41}$$

where the individual terms can be obtained from $K'(\boldsymbol{a})$ themselves:

$$U(\boldsymbol{c}) = \sum_{\boldsymbol{a} \subseteq \boldsymbol{c}} (-1)^{c-a} K'(\boldsymbol{a}). \tag{4.42}$$

Clearly, the sets $\boldsymbol{a}$ and $\boldsymbol{c}$ must include the pair of cells in question, and $a(c)$ is the number of cells in the cluster $\boldsymbol{a}(\boldsymbol{c})$.

An approximation could be made by cutting the sum in (4.41) off at some point $\boldsymbol{d}$, but this would just be $K'(\boldsymbol{d})$. To be more systematic, choose $\boldsymbol{c}$ to be members of a class γ of equivalent sets and approximate by cutting the class summation off at some point δ:

$$K'(\delta) \equiv \sum_{\gamma \subseteq \delta} \sum_{\boldsymbol{c} \in \gamma} U(\boldsymbol{c}). \tag{4.43}$$

Collecting (4.42) into (4.43), $K'(\delta)$ is expressible as a sum of some combinatorial coefficient times $K'(\boldsymbol{a})$.

At the level of a seven-cell approximation, computer analysis shows agreements of 3% and 1% for ν and η respectively. Linear extrapolation from the fixed point to the K axis gives K_c to better than 0·5%. Convergence properties are also quite satisfactory, much better than the finite lattice method (Subbarao 1975) and the straightforward cumulant expansion.

4.2.3. Variational methods. Kadanoff (1975), the instigator of these methods, considers the problem of approximations to the exact recursion relations and derives upper and lower bounds for the errors thus introduced in calculating the *free energy*, G. Although directly measurable quantities are derivatives of G, which are not in principle bounded by the bounds on G, it is hoped that 'pathological' behaviour is absent and good results will appear nevertheless. This hope is in fact realised in a comprehensive analysis (Kadanoff *et al* 1976) in which excellent results are found, in $d = 2, 3, 4$, for many properties (see also Katz *et al* (1977) for interesting extensions of this approach to other values of d). We present here the general principles, one of the simplest realisations for the Ising model, and some of the results.

The free energy of a system (again in units of kT) is defined via the partition function (4.3) by

$$G_s\{\mathscr{H}\} = -\ln \sum_{\{s\}} \exp(-\mathscr{H}(s)). \tag{4.44}$$

The subscript s and curly brackets show the functional character of G. The RG transformation (4.1) can be written in such a form if we first use the properties of $W(\sigma, s)$ to rewrite it as

$$W(\sigma, s) = \exp[-\hat{R}(\sigma, s) + U(s)] \tag{4.45}$$

where $\hat{R}$ is an arbitrary (real) function of σ and s and $U(s)$ plays the role of normalisation of W (cf (4.2)):

$$U(s) = G_\sigma\{\hat{R}\}. \tag{4.46}$$

Following the notation of Kadanoff *et al* (1976) we use the hat symbol (^) to denote functions of both σ and s. Defining

$$\hat{\mathscr{H}}(\sigma, s) = \mathscr{H}(s) - U(s) + \hat{R}(\sigma, s) \tag{4.47}$$

we see that (4.1) can be cast into the form

$$\mathscr{H}'(\sigma) = G_s\{\hat{\mathscr{H}}\}. \tag{4.48}$$

Furthermore, we obviously can retrieve $\mathscr{H}(s)$ by

$$\mathscr{H}(s) = G_\sigma\{\hat{\mathscr{H}}\} \tag{4.49}$$

so that the equality of the free energies of the σ and s systems is clear:

$$G_\sigma\{\mathscr{H}'\} = G_{\sigma,\,s}\{\hat{\mathscr{H}}\} = G_{s,\,\sigma}\{\hat{\mathscr{H}}\} = G_s\{\mathscr{H}\} \tag{4.50}$$

where

$$G_{\sigma,\,s}\{\hat{\mathscr{H}}\} \equiv G_\sigma\{G_s\{\hat{\mathscr{H}}\}\}. \tag{4.51}$$

Since $G_{\sigma,\,s}$ is nothing but $-\ln \Sigma_{\{\sigma,s\}}\exp(\hat{\mathscr{H}})$, it is the same as $G_{s,\,\sigma}$.

Using this formalism, we proceed to analyse approximations $\hat{\mathscr{H}}^A$, of $\hat{\mathscr{H}}$ and derive conditions for the positive or negative definiteness of the difference:

$$\Delta G \equiv G_{\sigma,\,s}\{\hat{\mathscr{H}}\} - G_{\sigma,\,s}\{\hat{\mathscr{H}}^A\}. \tag{4.52}$$

The methods are standard: introduce the interpolating 'Hamiltonian'

$$\hat{\mathscr{H}}_\lambda = \hat{\mathscr{H}}^A + \lambda\hat{V} \tag{4.53}$$

where

$$\hat{V} \equiv \hat{\mathscr{H}} - \hat{\mathscr{H}}^A \tag{4.54}$$

so that

$$\Delta G = \int_0^1 \mathrm{d}\lambda(\partial_\lambda G_{\sigma,\,s}\{\hat{\mathscr{H}}_\lambda\}). \tag{4.55}$$

Note also that

$$\partial_\lambda G_{\sigma,\,s}\{\hat{\mathscr{H}}_\lambda\} = \langle\hat{V}\rangle_\lambda \equiv \Big[\sum_{\{\sigma,\,s\}} \hat{V}\exp(-\hat{\mathscr{H}}_\lambda)\Big]\Big[\sum_{\{\sigma,\,s\}} \exp(-\hat{\mathscr{H}}_\lambda)\Big]^{-1} \tag{4.56}$$

and

$$\partial_\lambda{}^2 G_{\sigma,\,s}\{\hat{\mathscr{H}}_\lambda\} = -\langle[\hat{V} - \langle\hat{V}\rangle_\lambda]^2\rangle_\lambda \leqslant 0. \tag{4.57}$$

Integrating (4.55) by parts once leads to

$$\Delta G = \langle\hat{V}\rangle_{\lambda=1} - \int_0^1 \mathrm{d}\lambda\lambda(\partial_\lambda{}^2 G). \tag{4.58}$$

Adding $\partial G/\partial\lambda|_0{}^1$ to the second term and subtracting it from the first gives

$$\Delta G = \langle\hat{V}\rangle_{\lambda=0} + \int_0^1 \mathrm{d}\lambda(1-\lambda)(\partial_\lambda{}^2 G). \tag{4.59}$$

Since $\partial_\lambda{}^2 G$ is negative definite, if we choose an approximation satisfying $\langle\hat{V}\rangle_{\lambda=1} = 0$ ($\langle\hat{V}\rangle_{\lambda=0} = 0$) then the approximate free energy will be a lower (upper) bound to the exact one.

The first-order cumulant of Niemeijer and van Leeuwen (1973), for example, provides an upper bound. To see this, we need to consider only one intra-cell bond, of which $\hat{V}$ ($= V$) is a sum. First perform $\Sigma_{\{\sigma\}}$, then

$$\langle Ks_{i,\,\mu}s_{j,\,\nu}\rangle_{\lambda=0} \propto \sum_{\{s\}} Ks_\mu s_\nu \exp(-\mathscr{H}_0) \propto \sum_{\substack{\{s\}\text{ in}\\ \text{cells }\mu,\,\nu}} Ks_\mu s_\nu \exp(h_0(s,\,\mu) + h_0(s,\,\nu)) \tag{4.60}$$

which is zero for $H = 0$ or positive definite for $H \neq 0$. In the latter case, (4.59) is still negative definite. (Note that the average here (4.56) is over all $\{\sigma, s\}$, not just $\{\bar{s}\}$ as in (4.35).)

Noting that this upper bound does not provide particularly good results, Kadanoff explored the possibility of the other bound. Of course, in general, it is difficult to choose an approximation such that the average of $\hat{V}$ under the *exact* $\hat{\mathscr{H}}$ vanishes ($\langle\hat{V}\rangle_{\lambda=1} = 0$). Symmetry properties of $\hat{\mathscr{H}}$ are exploited and $\hat{\mathscr{H}}^A$ is suitably chosen

so that $\hat{V}$ has properties 'opposite' to $\mathscr{H}$. Then $\langle \hat{V} \rangle$ is automatically zero. To illustrate this method we work with a specific scheme on the $d=2$ square lattice. Generalisations may be found in Kadanoff *et al* (1976).

The model under consideration will be defined by the interaction Hamiltonian for spins on a square lattice:

$$-\mathscr{H}(s) = -\sum_q v(s, q) = \sum_q \left[f + K\left(\sum_{i \neq j} s_{i,q} s_{j,q} \right) + N \prod_i s_{i,q} \right] \tag{4.61}$$

where q labels a square of unit lattice spacing squared (also called a plaquette) and $i, j = 1, \ldots, 4$ are sites at the corners of a square. Σ_q denotes the sum over *all* plaquettes. Since a site belongs to four plaquettes, each site can be labelled in four ways. Each pair of nearest neighbours appears in two plaquettes. Thus (4.61) corresponds to nearest-neighbour interactions of strength $2K$, next-nearest-neighbour of K and a four-spin one of N. Note that a constant is included in $\mathscr{H}$. In Kadanoff's lower bound approximation only these types of couplings are required. Further, the specific choice of $v(s, q)$ in (4.61) is invariant under permutations of the spins in the plaquette. This property is reproduced under RG transformation by the choice of R in (4.62) below. Therefore the space of interactions (f, K, N) is *closed* in this approximation. In practice, such a Hamiltonian can be obtained from the standard Ising one ($\Sigma_{nn} K_I ss$) by an exact Wilson-like decimation (cf (4.18)–(4.20)), giving

$$\exp(-v(s, q)) = 2 \cosh K_I \left(\sum_{i=1}^{4} s_{i,q} \right).$$

Next we define the transformation with a parameter p:

$$\hat{R}(\sigma, s) \equiv p \sum_\mu \sigma_\mu (s_{1,\mu} + s_{2,\mu} + s_{3,\mu} + s_{4,\mu}) \tag{4.62}$$

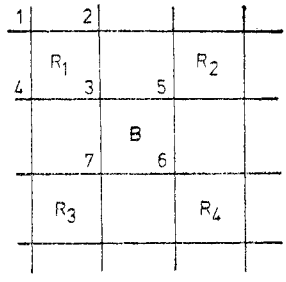

Figure 10. Possible configuration of red (R) and blue (B) cells on a square lattice for the variational method.

where, with reference to figure 10, μ labels the 'red' cells and the sites in each square are relabelled by the pair (i, μ). $U(s)$ can be found via (4.46), a sum over μ of (μ suppressed) terms

$$u(s, \mu) = u_f + u_K \sum_{i \neq j}^{4} s_i s_j + u_N s_1 s_2 s_3 s_4. \tag{4.63}$$

The coefficients are functions of p:

$$u_f = \ln 2 + l_2 + l_4 \qquad u_K = l_4 \qquad u_N = l_4 - l_2 \tag{4.64}$$

where

$$l_2 = \tfrac{1}{2} \ln \cosh 2p \qquad \text{and} \qquad l_4 = \tfrac{1}{8} \ln \cosh 4p. \tag{4.65}$$

Now $\mathscr{H}$ takes the following form:

$$\mathscr{H}(\sigma, s)=\sum_{q} v(s, q)+p\sum_{\mu}(\sigma\sum s)-\sum_{\mu} u(s, \mu) \tag{4.66}$$

which we approximate by

$$\mathscr{H}^{A}(\sigma, s)=\sum_{\beta}\left(4v-u+p\sum_{i} s_{i,\beta}\sigma_{\mu_i}\right) \tag{4.67}$$

where the summation is taken over 'blue' cells only, and $\sigma_{\mu i}$ is the cell spin in the red cell to which $s_{i,\beta}$ belongs. With the example in figure 10 this means $p(\sigma_1 s_3+\sigma_2 s_5+\sigma_3 s_6+\sigma_4 s_7)$ (R_μ contains σ_μ). Since a site appears once only in both (4.62) and the last term in (4.67), they are the same.

The choice of $\mathscr{H}^A$ is clearly motivated by the factorisable property of $\exp(-\mathscr{H}^A)$ into a product over β giving a closed RG transformation. Next, we analyse $\hat{V}=\mathscr{H}-\mathscr{H}^A$. Since $\Sigma_q=\Sigma_\mu+\Sigma_\beta+\Sigma_w$, where the last sum is over the 'white' (blank in figure 10) cells,

$$\hat{V}=\sum_{\mu}(v-u)-\sum_{\beta}(v-u)+\sum_{w} v-2\sum_{\beta} v. \tag{4.68}$$

Note further that $\hat{V}$ is independent of σ so that the $\{\sigma\}$ average in $\langle\hat{V}\rangle$ is trivial and we get

$$\langle\hat{V}\rangle_{\lambda=1}\propto\sum_{\{s\}}\hat{V}\exp(-\mathscr{H}(s)). \tag{4.69}$$

Now $\mathscr{H}(s)$ is symmetric under the interchange of 'red', 'white' and 'blue' cells so that the contributions to (4.69) from the μ, β and w cells in $\hat{V}$ are indistinguishable and add up to zero. This is guaranteed when we take the form (4.67), simply shunting all interactions in cells into the blue cells. Since this is true for any p, we can choose p to maximise G.

Before quoting some results of Kadanoff, we give a summary of the procedure. Using (4.67) and (4.48), we get the RG equations:

$$f'=f'(p; f, K, N) \tag{4.70(a)}$$

$$K'=K'(p; K, N) \tag{4.70(b)}$$

$$N'=N'(p; K, N) \tag{4.70(c)}$$

with which one can find the fixed point $\mathscr{H}^*(f^*, K^*, N^*)$ for each p. Now the free energy G calculated with this Hamiltonian will depend on p through $\mathscr{H}^*$ so that the stationary condition on G reads

$$0=\frac{\mathrm{d}G}{\mathrm{d}p}=\frac{\partial G}{\partial f^*}\frac{\partial f^*}{\partial p}+\frac{\partial G}{\partial K^*}\frac{\partial K^*}{\partial p}+\frac{\partial G}{\partial N^*}\frac{\partial N^*}{\partial p}. \tag{4.71}$$

The second factor of each term is readily obtained from (4.70), while the first factors are proportional to the expectation values (over $\mathscr{H}^*$) of the operators 1, $\Sigma_{i\neq j} s_{i,q}s_{j,q}$, and $\Pi_i s_{i,q}$ respectively. Alternatively, they are the elements of the eigenvector, with eigenvalue 4, of the stability matrix at $\mathscr{H}^*$, again obtainable from (4.70). The critical indices are also calculated from that matrix. The interested reader will enjoy the exercise of following this programme in one dimension (where it is exact) and on a triangular lattice, where analytic calculation (only nearest-neighbour interactions are required) gives results good to around 10%. In the square $2d$ model above, $\mathscr{H}^*$ is found numerically for a range of p and there exists a $\bar{p}$ ($=0{\cdot}766$) for

which (4.71) holds. Furthermore, the $d=3$ and 4 cases are also easily tractable on a computer. With p^* held fixed at the best value for the fixed point H^*, diagonalisation of the stability matrix gives the remarkable exponents in table 1. (Strictly the best p will vary as $\mathscr{H}$ is perturbed from $\mathscr{H}^*$; when this is allowed results are surprisingly not quite as good.) Results for free energy, specific heat and spontaneous magnetisation are similarly excellent.

Table 1. Values of exponents for the Ising model from the variational method (Kadanoff *et al* 1976). Results in brackets are from: $d=2$ (exact solution); $d=3$ (estimates from series expansions); $d=4$ (mean-field results) see §5.

Critical index	Dimensionality d		
	2	3	4
α	0·0017	0·1132	0·035
	(0)	(0·08 ± 0·04)	(0)
β	0·12457	0·3243	0·5033
	(0·125)	(0·3125 ± 0·005)	(0·5)
γ	1·7491	1·238	0·958
	(1·75)	(1·250 ± 0·005)	(1·0)
δ	15·040	4·818	2·90
	(15)	(5·0 ± 0·2)	(3)

However, we must finish with a word of caution. Burkhardt (1976) and Knops (1976) analysed the RG trajectories (for the $d=2$ case) in the general coupling space of unrelated nearest- and next-nearest-neighbour coupling, rather than the above subspace $\mathscr{S}$ of permutation-symmetric couplings. The conclusion is that for $p>0{\cdot}741$, the above $\mathscr{H}^*$ is not stable with respect to perturbations out of $\mathscr{S}$. In fact more stable fixed points (not in $\mathscr{S}$) do exist. Furthermore, if one starts the recursion relations from the standard (nearest neighbour only) Ising model rather than after one Wilson decimation (which brings $\mathscr{H}$ into $\mathscr{S}$), then only these new fixed points can be reached. The exponents associated with the new fixed points are still accurate to around 2%. The very bad feature, however, is that all these fixed points are close together and have a correction to scaling exponent which is very close to zero and produces very slow transient terms which are not at all in agreement with corrections to scaling in the exact solution of the Ising model. One may speculate that this effect is perhaps the shadow (after the shunting approximation) of the existence of redundant variables. As yet there exists no convincing explanation of this defect of the variational methods.

5. Field theoretic methods

In this section we turn to the problem of formulating the RG for field theory models involving a spin density rather than a set of spins at the sites of a lattice. Such field models are *a priori* no less physical than lattice models if we are interested in critical behaviour which is governed by fluctuations of all length scales. There are strong connections in fact between lattice and field models and we indicate in §5.3.2 how the Ising model can be formulated as a field theory. This reformulation of the Ising model can be used as a vehicle to illustrate the kinds of interactions

which the spin density may have, but we prefer to work to this goal from Landau theory as a starting point.

In this section we shall therefore discuss Landau theory and use it to motivate a statistical mechanical model of a field representing a spin density. The systematic approximation scheme in this model is the Feynman graph expansion. The lowest-order term in this expansion gives Landau theory. We shall see by naive dimensional analysis how Landau theory breaks down in the critical region when the number of space dimensions d is less than 4; the higher-order terms in the Feynman graph expansion are infinitely larger in the critical region than those in Landau theory. We shall then outline the various RG methods in field models, indicating how they give rise to systematic corrections to Landau theory in powers of $\epsilon = 4 - d$ when d is less than 4.

Many reviews of the substantial and sophisticated developments of both the Feynman graph expansion and the RG in field theories already exist. Standard texts on the Feynman graph expansion are Bjorken and Drell (1965), Bogoliubov and Shirkov (1959) (in relativistic quantum theories) and Fetter and Walecka (1971) (many-body physics). Amit (1977) develops Feynman graphs with a specific view to applications to critical phenomena. Elements of the Feynman graph expansion in Wilson recursion formulae are contained in Wilson and Kogut (1974), Wilson (1975) and Aharony (1976b). For those already conversant with Feynman graph methods an authoritative review is given by Brézin *et al* (1976a); see also Brézin (1975b), Wallace (1976), Weinberg (1976), Delbourgo (1976) and Macfarlane (1977). In this review we do not set up the mechanics of the Feynman graph expansion; we wish simply to give the reader some insight into the ϵ expansion and sufficient understanding to appreciate the applications summarised in the next section. Anyone aiming to do calculations using these methods must turn to the other reviews given above.

5.1. Landau theory

Consider the Helmholtz thermodynamic potential $A(T, M)$ for some magnetic system. If we know the form of $A(T, M)$, we can obtain all of the thermodynamic properties of the system as discussed in §3.2.1. For example, the external magnetic field, and hence the equation of state, is given by

$$H = \frac{\partial A(T, M)}{\partial M} \tag{5.1}$$

and the susceptibility $\chi = \partial M / \partial H$ by

$$\chi^{-1} = \frac{\partial H}{\partial M} = \frac{\partial^2 A(T, M)}{(\partial M)^2} \tag{5.2}$$

etc.

In Landau theory one makes a guess for the form of A based on the assumption that it can be expanded analytically in $T - T_c$ and M. In the critical region where both of these quantities are small only the dominant low-order terms are retained. Specifically we write

$$A(T, M) = a_0(T) + a_1(T)M^2 + a_2(T)M^4 + \mathrm{O}(M^6)$$

where odd powers in M vanish because symmetry under $M \to -M$ is always assumed.

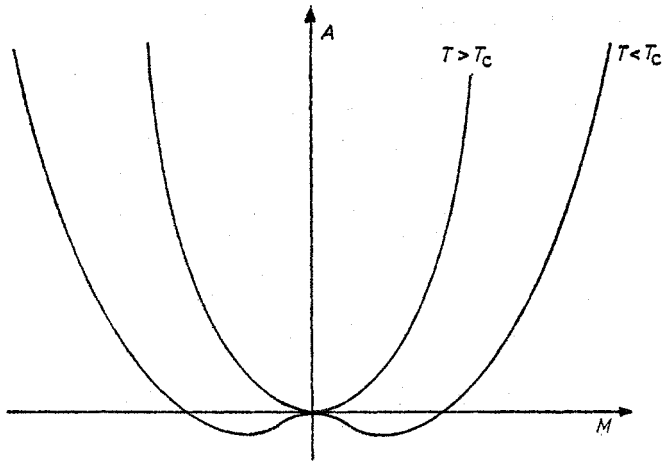

Figure 11. Qualitative form of thermodynamic potential for $a_1 > 0$ $(T > T_c)$ and $a_1 < 0$ $(T < T_c)$.

The critical temperature is recognised by noting that provided $a_2(T) > 0$, the form of $A(T, M)$ changes from (a) to (b) in figure 11 as $a_1(T)$ changes from positive to negative. When $a_1(T) < 0$, $\partial A/\partial M = 0$ has two non-trivial roots for M corresponding to the existence of two (up/down) values of spontaneous magnetisation. Therefore the onset of spontaneous magnetisation occurs when $a_1(T)$ changes sign, i.e.

$$a_1(T) = a(T - T_c) + O((T - T_c)^2)$$

where a is a positive constant.

Retaining only the leading terms we arrive at

$$A(T, M) = a_0(T) + \tfrac{1}{2}(T - T_c)M^2 + \tfrac{1}{4}M^4 \tag{5.3}$$

where the constants a and $a_2(T_c)$ have been absorbed into the scales of $T - T_c$ and M. This form of the free energy gives the Landau theory of phase transitions. Up to the neglect of irrelevant M^6, $(T - T_c)M^2$, etc, terms it is typical of the forms of the free energy obtained in most 'self-consistent' approximations, such as the mean field or Curie–Weiss theory of phase transitions (Stanley 1971, chap 5 and 6).

From equations (5.1) and (5.3) we immediately obtain the equation of state

$$H = (T - T_c)M + M^3$$

or

$$H/M^3 = (T - T_c)/M^2 + 1. \tag{5.4}$$

This exhibits the scaling form of

$$H/M^\delta = h((T - T_c)/M^{1/\beta})$$

with the identification

$$\delta = 3 \qquad \beta = \tfrac{1}{2} \qquad h(x) = x + 1. \tag{5.5}$$

The exponents are in reasonable qualitative agreement with the experimental $(d = 3)$ values $\delta \simeq 4{\cdot}5$, $\beta \simeq 0{\cdot}35$, and a typical form for $h(x)$ is shown in figure 12.

Similarly from (5.2) we have

$$\chi = \frac{1}{T - T_c + 3M^2}. \tag{5.6}$$

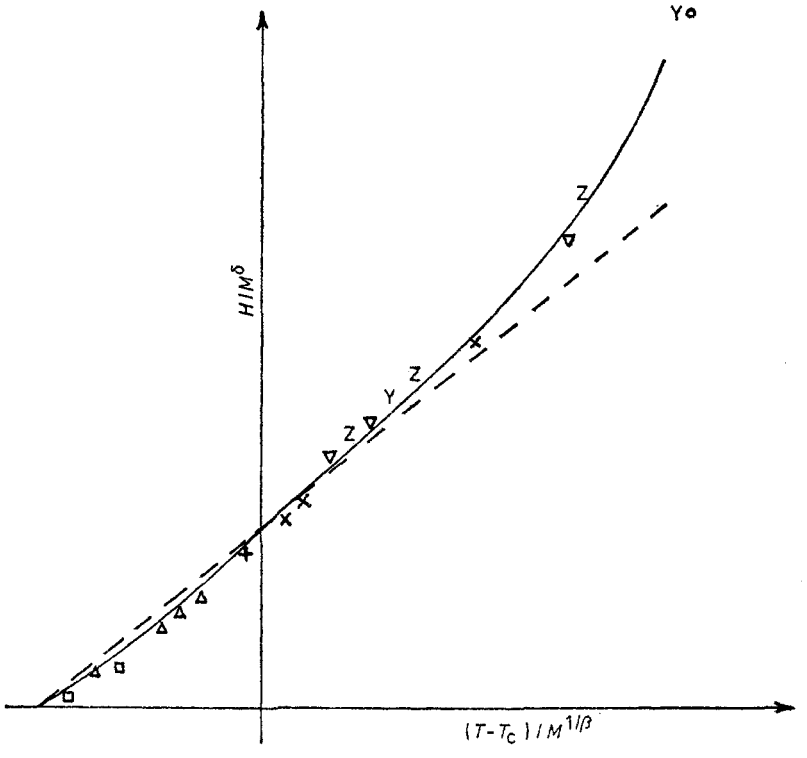

Figure 12. Schematic illustration of the scaling form of the equation of state. This is a simplification of the data of Kouvel and Comly (1968) on nickel, with the full curve fit in Vicentini-Missoni *et al* (1969). The straight broken line is the mean field equation of state, appropriately normalised. □, 622·5 K; △, 625 K; +, 627·5 K; ×, 630 K; ▽, 632·5 K; Z, 635 K; Y, 640 K; ○, 645 K. $\delta=4{\cdot}44$, $\beta=0{\cdot}373$, $T_c=627{\cdot}4$ K.

Then, for example, if we consider $T>T_c$ and $H=0$, so that $M=0$, we can identify $\chi=(T-T_c)^{-\gamma}$ with $\gamma=1$, in agreement with the scaling law $\gamma=\beta(\delta-1)$ (equation (3.67)) and not far from the experimental range $\gamma\sim 1{\cdot}25$–$1{\cdot}37$ for various kinds of magnetic systems in three dimensions.

The Landau theory above may be extended to include a treatment of correlation functions. If smooth spatial variations in the mean magnetisation are allowed, then the generalisation of equation (5.3) looks like

$$A(T, M)=\int \mathrm{d}^d x[a_0(T)+\tfrac{1}{2}\zeta(\nabla M(x))^2+\tfrac{1}{2}(T-T_c)M^2(x)+\tfrac{1}{4}M^4(x)]. \qquad (5.7)$$

In the same spirit which led to equation (5.3), we neglect here higher-order terms in $\nabla M(x)$, assuming $M(x)$ to be only slowly varying. The correlation function $G(x, y)$ gives the change in $M(x)$ due to a small change in the applied field $H(y)$:

$$G(x, y)=\frac{\delta M(x)}{\delta H(y)}. \qquad (5.8)$$

It is the inverse (in a matrix sense with x and y as indices) of $\delta H(y)/\delta M(x)$. Now from (5.7):

$$H(y)\equiv\frac{\delta A}{\delta M(y)}=-\zeta\nabla^2 M(y)+(T-T_c)M(y)+M^3(y)$$

and

$$\frac{\delta H(y)}{\delta M(x)} = [-\zeta \nabla_y{}^2 + (T - T_c) + 3M^2(y)]\delta^d(x-y).$$

Taking the Fourier transform of this expression for $M(y) \equiv M$, a constant, gives

$$\tilde{G}(q) \equiv \int \mathrm{d}^d x \exp(iqx) G(x, 0)$$

$$= \frac{1}{\zeta q^2 + (T - T_c) + 3M^2}. \tag{5.9}$$

This is the familiar Ornstein–Zoernicke form (see Stanley 1971, chap 7) of the correlation function. It exhibits all the scaling properties discussed in §3.2.2, with

$$\tilde{G}(0) = \chi \tag{5.10}$$

(see equation (5.6)), and

$$\tilde{G}(q) = \frac{1}{\zeta(q^2 + \xi^{-2})} \tag{5.11}$$

where the correlation length ξ is given by

$$\xi^2(T, M) = \frac{\zeta^{-1}}{(T - T_c) + 3M^2} \tag{5.12}$$

so that

$$\eta = 0 \qquad \nu = \tfrac{1}{2}. \tag{5.13}$$

See Brézin (1975b) for further discussion.

5.2. Statistical mechanics of fields

5.2.1. Basic formalism. Although Landau theory gives a reasonable qualitative description of a three-dimensional phase transition, it is certainly inadequate quantitatively for most systems. One may underline the inadequacy by pointing out that Landau theory is really *too* universal. One obtains the same exponents and equation of state (5.5), independently of the dimensions of space d, or the number of components of the magnetisation. Critical exponents in, for example, uniaxial (Ising-like) systems certainly do depend on d, e.g. $\beta = \frac{1}{8}$, $\gamma = 7/4$ for the exactly soluble, two-dimensional Ising model.

The origin of this inadequacy of Landau theory lies in its neglect of fluctuations. In field models we study the statistical mechanics of a field $\phi(x)$ whose average value gives the magnetisation. The prototype Hamiltonian has a form reminiscent of the free energy in Landau theory:

$$\mathscr{H} = \int \mathrm{d}^d x \left(\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2} r_0(T)\phi^2(x) + \frac{1}{4!} u_0 \phi^4(x) \right). \tag{5.14}$$

The local interactions $\phi^2(x)$ and $\phi^4(x)$ have a form similar to (5.7). The essential temperature dependence is contained in $r_0(T)$. It is sufficient to assume that r_0 is an analytic function of T, behaving, in the neighbourhood of T_c, like

$$r_0(T) - r_0(T_c) = a(T - T_c) + \mathrm{O}((T - T_c)^2) \tag{5.15}$$

as in (5.3). The $(\nabla\phi)^2$ term ensures that short-wavelength fluctuations (magnetisation densities varying rapidly in space) have a high energy; the scale of ϕ is con-

ventionally chosen so that this term has coefficient $\frac{1}{2}$. Such a 'kinetic' term with positive coefficient mimics interactions which tend to align the spins, in the sense that its effect is to favour states with uniform spin density. Unlike Landau theory, the assumptions of analyticity in ϕ and T in equation (5.14) are now justified because we are considering a *microscopic* model for the system, cf the analytic expansion of the potential for atoms in a crystal, in powers of the displacements of the atoms from equilibrium.

We shall use the Hamiltonian (5.14) in the Boltzmann factor $\exp(-\mathscr{H})$ (i.e. the conventional $1/kT$ is already absorbed into $\mathscr{H}$ for simplicity).

The presence of an external field $H(x)$ correspond to adding a term

$$-\int \mathrm{d}^d x H(x)\phi(x) \tag{5.16}$$

to the Hamiltonian.

We stress that although qualitative considerations of symmetry and range of interactions place strong restrictions on the form of the Hamiltonian, the ultimate justification for the use of a Hamiltonian like (5.14) rests in showing that other possible interactions, $\phi^6(x)$, $\phi^2\nabla^2\phi^2$, etc, are *irrelevant*, in the sense that they do not change the critical behaviour obtained from (5.14). This philosophy underlies all applications of the renormalisation group.

Given a Hamiltonian, correlation functions are calculated by averaging the Boltzmann factor $\exp(-\mathscr{H})$ over all configurations of the field. We write (cf equation (2.4))

$$\langle\phi(x_1)\ldots\phi(x_N)\rangle = \frac{\int \mathrm{D}\phi\ \phi(x_1)\ldots\phi(x_N)\exp(-\mathscr{H})}{\int \mathrm{D}\phi \exp(-\mathscr{H})} \tag{5.17}$$

where $\int \mathrm{D}\phi$ denotes the functional integral—the 'sum' over all possible field configurations $\phi(x)$.

The functional integral is discussed in many articles (see e.g. the field theory reviews mentioned in §5.1). A rigorous mathematical treatment requires considerable sophistication (see, for example, Glimm and Jaffe 1976). In practice, this problem is simply side-stepped by setting up a perturbation theory for expressions such as (5.17), which is called the Feynman graph expansion. This expansion forms the basic tool in the use of the renormalisation group in field models.

5.2.2. Landau theory as a zeroth-order approximation. Landau theory appears as the zeroth-order approximation in the Feynman graph expansion. To illustrate this consider the Gibbs free energy, given by the partition function in the denominator of (5.17):

$$\exp(-G) = \int \mathrm{D}\phi \exp(-\mathscr{H}) \tag{5.18}$$

where we include an external field term (5.16) in $\mathscr{H}$. The basis of the Feynman graph expansion is the evaluation of (5.18) by the method of steepest descent, in which the 'integrand' $\exp(-\mathscr{H})$ is approximated in zeroth order by its maximum value, i.e. with the value of ϕ which minimises $\mathscr{H}$.

In order to do this, one writes

$$\phi(x) = \phi_0(x) + \psi(x) \tag{5.19}$$

where $\phi_0(x)$ is the value of $\phi(x)$ which minimises the Hamiltonian, i.e.

$$\left.\frac{\delta\mathscr{H}}{\delta\phi}\right|_{\phi=\phi_0} = 0. \tag{5.20}$$

In principle ϕ_0 may depend on x, but we shall consider only the case where the external field H and ϕ_0 are x-independent. In the lowest-order approximation of the Feynman graph expansion we shall see that ϕ_0 corresponds to the expectation value of the field and hence we are considering the case of uniform external field and magnetisation. In this case the $(\nabla\phi)^2$ term in $\mathscr{H}$ does not enter into (5.20), which therefore takes the form

$$H = r_0(T)\phi_0 + \tfrac{1}{6}u_0\phi_0^3. \tag{5.21}$$

Translating the integration variable ϕ in equation (5.18) according to (5.19) we find

$$\exp(-G) = \exp\left[-\int \mathrm{d}^d x\left(\tfrac{1}{2}r_0\phi_0^2 + \frac{1}{4!}u_0\phi_0^2 - H\phi_0\right)\right]$$
$$\times \int \mathrm{D}\,\psi \exp\left[-\left(\int_x \int_y \frac{1}{2!}\frac{\delta^2\mathscr{H}}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_0}\psi(x)\psi(y) + \mathrm{O}(\psi^3)\right)\right]. \tag{5.22}$$

The effect of the translation (5.19) is to ensure that there is now no linear term in the integration variable $\psi(x)$ within the exponential. The Feynman graph expansion is obtained by systematic expansion of the anharmonic terms: $\exp(\mathrm{O}(\psi^3)) = 1 + \mathrm{O}(\psi^3)$. The zeroth-order approximation takes only the terms outside the integral, and therefore gives a Gibbs free energy density

$$g = \tfrac{1}{2}r_0\phi_0^2 + \frac{1}{4!}u_0\phi_0^4 - H\phi_0. \tag{5.23}$$

Similarly, one sees that ϕ_0 is to be identified with the magnetisation at the same level of approximation:

$$M \equiv \int \mathrm{D}\phi\ \phi \exp(-\mathscr{H}) / \int \mathrm{D}\phi \exp(-\mathscr{H})$$
$$= \phi_0 + \frac{\int \mathrm{D}\psi\ \psi \exp\left[-(2!)^{-1}\int_x \int_y (\delta^2\mathscr{H}/\delta\phi(x)\delta\phi(y)|_{\phi=\phi_0}\psi(x)\psi(y) + \mathrm{O}(\psi^3))\right]}{\int \mathrm{D}\psi \exp\left[-(2!)^{-1}\int_x \int_y (\delta^2\mathscr{H}/\delta\phi(x)\delta\phi(y))|_{\phi=\phi_0}\psi(x)\psi(y) + \mathrm{O}(\psi^3))\right]}$$
$$= \phi_0 \tag{5.24}$$

when terms of order ψ^3 in $\int \mathrm{D}\psi$ are neglected (the numerator is then odd under $\psi \to -\psi$).

Combining these results with the usual Legendre transform

$$A - g = HM$$

between Gibbs and Helmholtz free energy densities we see that we recover the results of Landau theory at zeroth order in the Feynman graph expansion:

$$A = \tfrac{1}{2}r_0(T)\,M^2 + \frac{1}{4!}u_0M^4$$

with the equation of state $H = r_0(T)M + u_0M^3/6$. As in Landau theory we can identify $r_0(T_c) = 0$ at this zeroth order so that we write, with a suitable choice of temperature scale,

$$r_0(T) = T - T_c. \tag{5.25}$$

A similar calculation gives, at the same order of approximation, the Ornstein–Zoernicke form

$$\tilde{G}(q)=\frac{1}{q^2+(T-T_c)+\frac{1}{2}u_0M^2} \tag{5.26}$$

for the generalised susceptibility, defined by

$$\tilde{G}(q)=\int \mathrm{d}^dx \exp(\mathrm{i}qx)\langle(\phi(x)-M)(\phi(0)-M)\rangle. \tag{5.27}$$

The derivation of expression (5.26) is the extension to a continuous set of variables of a result which holds for a discrete number of integration variables:

$$\frac{\int_{-\infty}^{\infty}\mathrm{d}y\ y_iy_j \exp(-\frac{1}{2}y_kA_{kl}y_l)}{\int_{-\infty}^{\infty}\mathrm{d}y \exp(-\frac{1}{2}y_kA_{kl}y_l)}=(A^{-1})_{ij}. \tag{5.28}$$

The denominator in (5.26) is just the 'matrix' $(\delta^2\mathscr{H}/\delta\phi(x)\delta\phi(y))|_{\phi=\phi_0}$ with $(-\nabla^2)$ in $\mathscr{H}$ converting to q^2 after Fourier transform. In fact, the Feynman graph expansion also has its origins in generalisations of (5.28).

5.3. Beyond Landau theory

5.3.1. The role of the cutoff and four dimensions. In order to go beyond Landau theory, an essential refinement of the form of the Hamiltonian is required. In expression (5.14), we include terms of second order in derivatives. This means that the energy of fluctuations of wavevector q increases as q^2, as q increases. Now, this is an appropriate form for low q; higher derivatives of the field just give higher powers of q, which are negligible if q is small. However, in order to obtain the free energy and correlation functions we must average over all possible functions $\phi(x)$, i.e. over fluctuations of all possible wavevectors. But the very existence of a lattice structure in the physics prohibits fluctuations with wavelengths *less* than the lattice spacing. This means that the q dependence in the energy must change drastically for q^2 near the edge of the Brillouin zone.

This 'memory' of the lattice spacing is called in field theory the cutoff, Λ. Its general role is to inhibit the influence of fluctuations of wavevector $q\gtrsim\Lambda$. It can be introduced in many ways. For example, with a so-called sharp cutoff the form of the Hamiltonian is left as in (5.14), but the average over the functions $\phi(x)$ in (5.17), for example, is understood to mean an average over functions whose Fourier components are less than Λ. In other words, the wavevectors permitted lie in a spherical Brillouin zone associated with a 'lattice spacing' $\sim\Lambda^{-1}$. Alternatively, one may add the coupling

$$\tfrac{1}{2}\int\Lambda^{-2}(\nabla^2\phi(x))^2\,\mathrm{d}^dx \tag{5.29}$$

to the Hamiltonian. The inhibition of fluctuations with $q\gtrsim\Lambda$ is now achieved because the energy of such fluctuations is large, increasing like q^4, and the Boltzmann factor $\exp(-\mathscr{H})$ is correspondingly small. Again Λ^{-1} has dimensions of length and is understood as the memory of the lattice.

The fact that Λ^{-2} appears as the natural scale of the dimensional coefficient of $(\nabla^2\phi(x))^2$ has a corollary; the generic (i.e. unexceptional) length scale of *all* dimensional coefficients in the microscopic Hamiltonian is the microscopic length scale Λ^{-1}. Alternatively stated, the Λ dependence of all dimensional parameters in $\mathscr{H}$ is determined by naive dimensional analysis. Specifically, since $\mathscr{H}$ appears in an

exponential and is dimensionless, then the dimension of the field ϕ is determined from $\int \mathrm{d}^d x(\nabla\phi)^2$ to be

$$[\phi] = [\Lambda]^{\frac{1}{2}(d-2)}. \tag{5.30}$$

Hence the coefficient u_{lm} of a general interaction:

$$u_{lm} \int \mathrm{d}^d x(\nabla^m \phi^l) \tag{5.31}$$

(symbolic notation for any power of ∇ on any of the ϕ) has dimension

$$[u_{lm}] = [\Lambda]^{d-m-\frac{1}{2}l(d-2)} \tag{5.32}$$

so that, e.g.

$$[r_0] = [T - T_c] = [\Lambda]^2 \qquad (m=0,\ l=2) \tag{5.33}$$

$$[u_0] = [\Lambda]^{4-d} \qquad (m=0,\ l=4) \tag{5.34}$$

and

$$[M] = [\phi] = [\Lambda]^{\frac{1}{2}(d-2)}. \tag{5.35}$$

With this preliminary, we can turn to the question of the nature of the corrections to Landau theory induced by higher-order terms in the Feynman graph expansion. A proper discussion clearly requires the systematic development of the Feynman graph expansion. We refer the reader particularly to the article by Brézin *et al* (1976a). Here we shall simply point out how dimensional analysis controls the validity of the Feynman graph expansion, and its zeroth-order approximation, Landau theory.

The Feynman graph expansion involves essentially the evaluation of the functional integral, $\int \mathrm{D}\phi$, as a power series in the anharmonic interactions. In effect, for the Hamiltonian (5.14) one obtains correlation functions, etc, as a power series in u_0. Now, at each order in u_0, additional powers of Λ^{4-d} appearing due to (5.34) must be cancelled by a quantity of the opposite dimension to ensure that these 'corrections' (or 'perturbations') have the same overall dimension as the lowest order. If only Λ^{d-4} plays this role, then there are no problems. However, there are other dimensional quantities, $T - T_c$, M and wavevector q, and we are interested in the critical behaviour where each of these quantities tends to zero. In this limit, the expansion in u_0 may therefore imply an expansion in the dimensionless quantities $(\Lambda^2/(T-T_c))^{\frac{1}{2}(4-d)}$ or $(\Lambda/M^{2/(d-2)})^{4-d}$ or $(\Lambda/q)^{4-d}$.

The importance of dimension four is now apparent. If $d<4$, these ratios become arbitrarily large in the critical region: perturbation theory breaks down and the Landau theory becomes invalid. If $d>4$, these ratios become arbitrarily small; perturbation theory and hence Landau theory is justified.

The actual mechanism of how these ratios appear is seen of course only in explicit calculations in the Feynman graph expansion. Suffice it to say that explicit calculation vindicates the above dimensional analysis, and shows moreover that when $d=4$, perturbation theory and Landau theory also break down because of the appearance of logarithms (of for example $\Lambda^2/(T-T_c)$ and Λ/q) at each order in perturbation theory.

This dimensional analysis also suggests why (when symmetry prohibits a ϕ^3 interaction) ϕ^4 is the most important interaction: all higher powers of ϕ or derivative couplings have a *smaller* power of Λ in the coupling constant, according to (5.32).

5.3.2. Lattice and field models. This subsection is a brief digression to indicate a precise connection between lattice and field models (Baker 1962, Siegert 1963).

Consider the Ising model in a space-varying external field H_i. All correlation functions can be obtained from the partition function

$$Z = \sum_{\{s\}} \exp\left(\tfrac{1}{2}s_i K_{ij} s_j\right) + H_i s_i \tag{5.36}$$

as discussed in §3.2.2. We have allowed arbitrary two-spin interactions K_{ij} and used the repeated index summation convention on the site labels. The identity

$$\exp\left(\tfrac{1}{2}s_i K_{ij} s_j\right) = C\left(\prod_i \int_{-\infty}^{\infty} \mathrm{d}\phi_i\right) \exp\left(-\tfrac{1}{2}\phi_i (K^{-1})_{ij}\phi_j + s_i\phi_i\right) \tag{5.37}$$

is obtained by completing the square in ϕ in the Gaussian integral; C is a normalising constant and K^{-1} is the matrix inverse of K. We can therefore rewrite (5.36) as

$$\begin{aligned} Z &= C\left(\prod_i \int_{-\infty}^{\infty} \mathrm{d}\phi_i\right) \exp\left[-\tfrac{1}{2}\phi_i (K^{-1})_{ij}\phi_j\right] \sum_{\{s\}} \exp\left[(H_i + \phi_i)s_i\right] \\ &= C\left(\prod_i \int_{-\infty}^{\infty} \mathrm{d}\phi_i\right) \exp\left[-\tfrac{1}{2}\phi_i (K^{-1})_{ij}\phi_j + \sum_i \ln\left(2\cosh\left(H_i + \phi_i\right)\right)\right]. \end{aligned} \tag{5.38}$$

Expression (5.38) is essentially in the form of a field theory, with the lattice providing a specific cutoff. The functional integral in equation (5.17) may be regarded as the continuum limit of $\Pi_i \int_{-\infty}^{\infty} \mathrm{d}\phi_i$. If K is short range, $\phi_i (K^{-1})_{ij}\phi_j$ is the analogue of terms $\int \mathrm{d}^d x a^{-d}$ [ϕ^2, $a^2(\nabla\phi)^2$, $a^4(\nabla^2\phi)^2$, etc] where $a(\sim \Lambda^{-1})$ is the lattice spacing. When ϕ is normalised so that the coefficient of $(\nabla\phi)^2$ is $\frac{1}{2}$, as in the Hamiltonian (5.14), all the features of (5.14) are recovered. The Λ dependence of coefficients is as in equation (5.32). The ϕ^2 term picks up a coefficient linear in $1/T$ where a constant term comes from $\phi K^{-1}\phi$ and the $1/T$ from the ϕ^2 term in the expansion of ln cosh ϕ (after the change of normalisation of the ϕ field and recalling $K \sim 1/T$). This coefficient is monotonic and analytic in T ($T > 0$) as required.

This discussion can obviously be extended to more general lattice models. It is possible also to set up a Feynman graph expansion for the expression (5.38). The zeroth-order term is the mean-field theory of the Ising model. The first-order term exhibits explicitly the breakdown of mean-field theory, dominated by the ϕ^4 interactions, as indicated in §5.3.1. For further discussion see Hubbard (1972a), Brézin (1975b) and Brézin *et al* (1976a).

5.4. Recursion formulae and ϵ expansion

The previous subsection shows how the problem of existence of two length scales, the lattice spacing and the correlation length, shows itself in field models. In this subsection we review the Wilson–Fisher recursion formulae (Wilson 1971a, b, Wilson and Fisher 1972) and the extent to which they provide a solution to the problem. The recursion formulae are in the same spirit as the block spin methods of §4. There the idea was to start from a system of a certain lattice spacing and transform it to an equivalent system with a larger lattice spacing. The analogue in field models is to start with a system with a sharp cutoff Λ; the Hamiltonian is as in equation (5.14) and only fluctuations with wavevector $q \leqslant \Lambda$ are considered. This system is then transformed to a system with a smaller cutoff Λ/b ($b > 1$) by doing the partial configuration sum over all fields with wavevector between Λ/b and Λ. There-

fore in the new system there are fluctuations with wavevector only between 0 and Λ/b.

This is easily formalised in terms of the Fourier component spin:

$$s(q)=\int \mathrm{d}^d x \exp(\mathrm{i}qx)\phi(x). \tag{5.39}$$

In terms of $s(q)$, the Hamiltonian (5.14) has the form

$$\begin{aligned}\mathscr{H}(s)=&\int_q^\Lambda (\tfrac{1}{2}q^2+\tfrac{1}{2}r_0(T))s(q)s(-q)\\ &+\frac{u_0}{4!}\int_{q_1}^\Lambda\int_{q_2}^\Lambda\int_{q_3}^\Lambda\int_{q_4}^\Lambda s(q_1)s(q_2)s(q_3)s(q_4)(2\pi)^d\delta^d\left(\sum_{i=1}^4 q_i\right)\end{aligned} \tag{5.40}$$

where

$$\int_q^\Lambda \equiv \int_{|q|<\Lambda} \mathrm{d}^d q(2\pi)^{-d}. \tag{5.41}$$

The Fourier-transformed correlation functions are given by

$$\begin{aligned}G^{(n)}(q)(2\pi)^d\delta^d\left(\sum_{i=1}^n q_i\right)&\equiv \prod_{i=1}^n \int \mathrm{d}x_i \exp(\mathrm{i}q_i x_i)\langle\phi(x_1)\ldots\phi(x_n)\rangle\\ &=\langle s(q_1)\ldots s(q_n)\rangle\\ &=\frac{\int \mathrm{D}s\, s(q_1)\ldots s(q_n)\exp(-\mathscr{H}(s))}{\int \mathrm{D}s\exp(-\mathscr{H}(s))}.\end{aligned} \tag{5.42}$$

The equivalence between the functional integrals (5.15) and (5.42) is obtained simply by regarding (5.39) as a linear change of integration variables. The momentum conserving δ functions are a consequence of the translation invariance of the theory.

In order to implement the above idea of doing a partial configuration sum, we write

$$s(q)=\begin{cases}\sigma(q) & |q|<\Lambda/b\\ \bar{\sigma}(q) & \Lambda/b\leqslant|q|<\Lambda\end{cases} \tag{5.43}$$

and define

$$\exp(-\tilde{\mathscr{H}}(\sigma))=\int \mathrm{D}\bar{\sigma}\exp(-\mathscr{H}(s)).$$

The new system $\tilde{\mathscr{H}}(\sigma)$ has fluctuations with wavevectors up to Λ/b, and it has the same partition function as $\mathscr{H}(s)$:

$$\begin{aligned}\int \mathrm{D}\sigma\exp(-\tilde{\mathscr{H}}(\sigma))&=\int \mathrm{D}\sigma\int \mathrm{D}\bar{\sigma}\exp(-\mathscr{H}(s))\\ &=\int \mathrm{D}s\exp(-\mathscr{H}(s))\end{aligned}$$

since the combined 'sum' on σ (which has $q<\Lambda/b$) and $\bar{\sigma}(\Lambda/b\leqslant q<\Lambda)$ constitutes the entire 'sum' on $s(q<\Lambda)$ (cf equation (2.1)).

The formalism so far parallels the decimation procedure of §2 (except that the effective Hamiltonian is labelled $\tilde{\mathscr{H}}$ and not $\mathscr{H}'$). Let us consider now the evaluation of $\int \mathrm{D}\bar{\sigma}$ in equation (5.44). It cannot be done exactly; the Wilson–Fisher (1972) recursion formula rests on a perturbative evaluation of $\int \mathrm{D}\bar{\sigma}$ by a generalisation of the conventional Feynman graph expansion. This is explained in some detail in Wilson and Kogut (1974) and Wilson (1975) (see also Hubbard 1973, Bissett 1973). One writes (in an obvious extension of the notation in (5.41)):

$$\mathscr{H}(s)=\tfrac{1}{2}\int_q^{\Lambda/b}(q^2+r_0(T))\sigma(q)\sigma(-q)+\tfrac{1}{2}\int_{q,\,\Lambda/b}^{\Lambda}(q^2+r_0(T))\bar{\sigma}(q)\bar{\sigma}(-q)$$
$$+\frac{u_0}{4!}\int_{q_1}^{\Lambda/b}\int_{q_2}^{\Lambda/b}\int_{q_3}^{\Lambda/b}\int_{q_4}^{\Lambda/b}\sigma(q_1)\sigma(q_2)\sigma(q_3)\sigma(q_4)(2\pi)^d\delta^d\left(\sum_{i=1}^{4}q_i\right)$$
$$+\frac{u_0}{4!}\left[\int_{q_1}^{\Lambda/b}\int_{q_2}^{\Lambda/b}\int_{q_3}^{\Lambda/b}\int_{q_4,\Lambda/b}^{\Lambda}\sigma(q_1)\sigma(q_2)\sigma(q_3)\bar{\sigma}(q_4)(2\pi)^d\delta^d\left(\sum_{i=1}^{4}q_i\right)\right.$$
$$\left.+\text{perms}+\ldots\right] \tag{5.44}$$

where 'perms' denotes permutations of 1, 2, 3 and 4 and $+\ldots$ denotes the terms of order $\bar{\sigma}^2$, $\bar{\sigma}^3$ and $\bar{\sigma}^4$. We can pull the part independent of the integration variable $\bar{\sigma}$ out of the integral, just as in equation (5.22), and obtain the zeroth-order form for $\tilde{\mathscr{H}}(\sigma)$:

$$\tilde{\mathscr{H}}(\sigma)=\tfrac{1}{2}\int_q^{\Lambda/b}(q^2+r_0(T))\sigma(q)\sigma(-q)$$
$$+\frac{u_0}{4!}\int_{q_1}^{\Lambda/b}\int_{q_2}^{\Lambda/b}\int_{q_3}^{\Lambda/b}\int_{q_4}^{\Lambda/b}\sigma(q_1)\sigma(q_2)\sigma(q_3)\sigma(q_4)(2\pi)^d\delta^d\left(\sum_i q_i\right)+\text{higher order.} \tag{5.45}$$

If u_0 were zero, expression (5.40) would give the exact σ dependence of $\tilde{\mathscr{H}}$. (The analysis of this so-called Gaussian model is given in Wilson and Kogut (1974).) The 'higher-order' terms evaluated as a power series in u_0 become increasingly more complicated. In general, new forms of interaction appear, just as in block spin methods. However, *provided that* u_0 *is small*, one can show that $\tilde{\mathscr{H}}(\sigma)$ has the form

$$\tilde{\mathscr{H}}(\sigma)=\tfrac{1}{2}\int_q^{\Lambda/b}(Aq^2+B)\sigma(q)\sigma(-q)$$
$$+\frac{C}{4!}\int_{q_1}^{\Lambda/b}\int_{q_2}^{\Lambda/b}\int_{q_3}^{\Lambda/b}\int_{q_4}^{\Lambda/b}\sigma(q_1)\sigma(q_2)\sigma(q_3)\sigma(q_4)(2\pi)^d\delta^d\left(\sum_{i=1}^{4}q_i\right)+\text{OT} \tag{5.46}$$

where

$$A=1+\mathrm{O}(u_0{}^2) \tag{5.47}$$

$$B=r_0-\tfrac{1}{2}ar_0u_0+cu_0+\mathrm{O}(u_0{}^2) \tag{5.48}$$

$$C=u_0(1-\tfrac{3}{2}au_0+\mathrm{O}(u_0{}^2)) \tag{5.49}$$

and OT represents other types of coupling, which are effectively higher order in u_0 than the results explicitly exhibited in (5.47)–(5.49). a and c are two constants which can be evaluated explicitly using the Feynman rules.

Equations (5.47)–(5.49) have been simplified (with hindsight, justifiably) to give the simplest possible equations to analyse. Note that the average over fluctuations with $\Lambda/b\leqslant|q|<\Lambda$ produces no embarrassing $(T-T_c)^{(d-4)/2}$ terms for example. Alternatively stated, the RG functions *are* calculable in perturbation theory.

The new effective Hamiltonian $\mathscr{H}'$ is obtained from $\tilde{\mathscr{H}}(\sigma)$ by (a) a rescaling of momenta

$$q'=bq \tag{5.50}$$

so that the cutoff in the q' variables is again Λ, and (b) a rescaling of fields

$$s'(q')=\zeta\sigma(q) \tag{5.51}$$

so that the gradient term in $\mathscr{H}'(s')$ has coefficient $\frac{1}{2}$, as in $\mathscr{H}(s)$:

$$\mathscr{H}'(s') = \tfrac{1}{2}\int_{q'}^{\Lambda} (q'^2 + r_0')s'(q')s'(-q') + \ldots . \tag{5.52}$$

These operations may be understood as exhausting the freedom of choice of scale of q' and s' in order to make $\mathscr{H}'(s')$ look as similar as possible to $\mathscr{H}(s)$: this choice will therefore optimise the likelihood of finding a fixed-point Hamiltonian, for which $\mathscr{H}' = \mathscr{H}$ $(= \mathscr{H}^*)$.

Combining these rescaling operations with equations (5.47)–(5.49), one finds readily

$$r_0' = b^2(r_0 - \tfrac{1}{2}ar_0u_0 + cu_0 + \mathrm{O}(u_0^2)) \tag{5.53}$$

$$u_0' = b^\epsilon u_0(1 - \tfrac{3}{2}au_0 + \mathrm{O}(u_0^2)) \tag{5.54}$$

$$\zeta = b^{(d-2)/2}(1 + \mathrm{O}(u_0^2)) \tag{5.55}$$

where $\epsilon = 4 - d$. The powers of b appearing initially on the right-hand sides of these equations arise from the rescalings (5.50) and (5.51) and are controlled by naive dimensions.

Let us see now how the ϵ expansion arises from equation (5.54). There is clearly the Gaussian fixed point $u_0^* = 0$. If we start with very small u_0, then for positive $\epsilon (d < 4)$, $u_0' \sim b^\epsilon u_0 > u_0$, i.e. the Gaussian fixed point is unstable to the ϕ^4 perturbations. However, there is a non-zero value of u_0 such that the growth from bu_0 is cancelled by the negative term of order u_0^2. Thus there is another fixed point, which is of order ϵ for ϵ small:

$$\begin{aligned} 1 &= b^\epsilon(1 - \tfrac{3}{2}au_0^* + \ldots) \\ &= (1 + \epsilon \ln b + \ldots)(1 - \tfrac{3}{2}au_0^* + \ldots) \end{aligned}$$

i.e.

$$au_0^* = \frac{2\epsilon \ln b}{3} + \mathrm{O}(\epsilon^2). \tag{5.56}$$

Equation (5.53) then gives a fixed point:

$$\begin{aligned} r_0^* &= -\frac{b^2}{b^2 - 1} cu_0^* + \mathrm{O}(u_0^{*2}) \\ &= -\frac{3cb^2 \ln b}{2a(b^2 - 1)} \epsilon + \mathrm{O}(\epsilon^2). \end{aligned} \tag{5.57}$$

The importance of ϵ as a small parameter should be clear. It enables us to find a non-trivial fixed point for $d < 4$ in which the anharmonic coupling is of order ϵ, and expansion in the anharmonic coupling gives a systematic expansion in ϵ. Higher orders in u_0 give higher orders in ϵ in equations (5.56) and (5.57).

Critical exponents are obtained from the stability matrix (see equation (3.4) and the subsequent discussion):

$$M_{ij} = \begin{pmatrix} b^2(1 - \frac{1}{3}\epsilon \ln b) & cb^2 + \mathrm{O}(\epsilon) \\ 0 & 1 - \epsilon \ln b \end{pmatrix} + \mathrm{O}(\epsilon^2). \tag{5.58}$$

The eigenvalues of this matrix are:

(i) $b^2(1 - \frac{1}{3}\epsilon \ln b) + \mathrm{O}(\epsilon^2)$. This is greater than 1 for ϵ small enough and represents the unstable perturbation proportional to $(T - T_c)$; for a given u_0, $r_0(T)$ must

be chosen equal to the special value $r_0(T_c)$ in order to ensure that there is no component along this eigenvector. According to equation (3.12), the exponent ν is given by

$$\nu = \frac{\ln b}{\ln [b^2(1-\frac{1}{3}\epsilon \ln b)]} + O(\epsilon^2)$$

$$= \tfrac{1}{2} + \tfrac{1}{12}\epsilon + O(\epsilon^2). \tag{5.59}$$

(ii) $1-\epsilon \ln b + O(\epsilon^2)$. This is less than 1 for ϵ small enough and represents a stable perturbation by an irrelevant interaction. According to equation (3.20), this produces corrections to the leading singular behaviour proportional to $(u_0-u_0^*)(T-T_c)^{\epsilon/2}+O(\epsilon^2)$. This is a so-called 'slow transient' effect. Note that if we chose $u_0=u_0^*$, then we can eliminate these transient terms. This is the basis of the Feynman graph method (Wilson 1972) for calculating critical exponents and scaling functions.

The fact that the change in field scale ζ (equation (5.51)) is determined by naive dimensional analysis to order ϵ^2 (equation (5.55)) implies $\eta = O(\epsilon^2)$ (equation (3.88) *et seq*).

We remark:

(i) The method can in principle be extended to higher orders in ϵ, but in general the 'other terms' in equation (5.52)—$O(s^6, q^2s^4$, etc)—have to be included and the calculations become correspondingly complicated. Critical exponents associated with an external field H can also be obtained (Hubbard 1972b). Values are tabulated in §6.1.

(ii) Although the principal exponents have mean-field values for $d>4$, correlation functions of many fields may develop non-mean-field singularities (see, for example, Fisher 1974b).

(iii) The limit $b \gg 1$ has been studied by Aharony (1973c) and Bruce *et al* (1974). They find that simplifications occur, and that the order ϵ^2 calculation may be performed by including *only* ϕ^4 interactions in this limit. Although it is not the most powerful method for obtaining ϵ expansion, this approach has been used for many other practical calculations, e.g. Priest and Lubensky (1976); see Aharony (1976b) for further references.

(iv) The limit $b \to 1$ is also interesting (Wegner and Houghton 1973). It corresponds to averaging over fluctuations with wavevectors in vanishingly thin shells. If we write $b=\exp \tau$, such contributions are proportional to τ and recursion formulae such as (5.53)–(5.55) become differential equations for a continuous family of effective coupling constants $u_0(\tau)$:

$$u_0' = \exp(\tau\epsilon)\, u_0 - \tfrac{3}{2}\tau\tilde{a}u_0^2 + O(u_0^3)$$

i.e.

$$\frac{du_0}{d\tau} = \epsilon u_0 - \tfrac{3}{2}\tilde{a}u_0^2 + O(u_0^3). \tag{5.60}$$

This is to be solved with initial conditions $u_0(0)=u_0$.

The structure of the equations changes significantly from the case above where b is arbitrary.

Firstly the analysis of scaling properties is much simpler using a differential form (5.60) rather than a recursion formula such as (5.54). For example, there is now no possibility of the oscillatory behaviour in, for example, equation (3.43).

The analogue of the general case (3.2) involves a family of differential equations, one equation for each coupling:

$$\frac{\mathrm{d}u_i}{\mathrm{d}\tau}=\beta_i(\boldsymbol{u}). \tag{5.61}$$

Fixed points are solutions of

$$\beta_i(u^*)=0 \tag{5.62}$$

and exponents y_a (equation (3.32)) are eigenvalues of the stability matrix

$$\frac{\partial\beta_i}{\partial u_j}(u^*). \tag{5.63}$$

For example, in $4-\epsilon$ dimensions, equation (5.60) gives

$$u_0{}^*=\frac{2\epsilon}{3\bar{a}}+\mathrm{O}(\epsilon^2) \tag{5.64}$$

and

$$(\partial\beta/\partial u_0)(u_0{}^*)\equiv\omega=-\epsilon+\mathrm{O}(\epsilon^2). \tag{5.65}$$

For u close to u^*, we have

$$\frac{\mathrm{d}(u_0(\tau)-u_0{}^*)}{\mathrm{d}\tau}=\omega(u_0(\tau)-u_0{}^*)+\mathrm{O}((u_0(\tau)-u_0{}^*)^2)$$

with the solution

$$u_0(\tau)-u_0{}^*\simeq\exp(\omega\tau)(u_0-u_0{}^*) \tag{5.66}$$

so that the fixed point (5.64) is stable since ω is negative for small ϵ. The exponent ω governs corrections to scaling, as discussed in §3.

Secondly the higher orders in u_0 in (5.60) come from Feynman graphs with more than one loop. These graphs involve integrations over a shell $\Lambda(1-\tau)<|k_i|<\Lambda$ for each independent loop momentum k_i. Thus each loop produces a factor τ and hence two and higher loops are negligible in the limit $\tau\to 0$ compared with the one-loop result (5.60). The ϕ^6 coupling, v_0 say, does appear in the equation for $\mathrm{d}u_0/\mathrm{d}\tau$. The differential equation for $v_0(\tau)$ involves the ϕ^8 couplings, etc. Thus in this thin-shell formulation one obtains a hierarchy of exact, coupled differential equations.

This method is not efficient for ϵ-expansion calculations but may be a vehicle for other approximation schemes—see (v) below.

The first appearance (historically) of a differential formalism (which does not in fact have the second feature above) is discussed in §5.5. The use of differential equations rather than the discrete transformations (3.2) is much more important than its meagre treatment here might suggest.

(v) All of the above schemes use a sharp cutoff—fluctuations with wavevectors $|q|>\Lambda$ are not permitted. This choice of cutoff is of course arbitrary and, in practice, must be handled carefully (Wegner and Houghton 1973). In the 'smooth cutoff' formalism (Wilson and Kogut 1974) one starts with a formalism similar to that in lattice models (equation (4.1)):

$$\exp(-\mathscr{H}_\tau(\phi'))=\int\mathrm{D}\phi W(\phi',\phi,\alpha_q(\tau))\exp(-\mathscr{H}(\phi)). \tag{5.67}$$

$\mathscr{H}(\phi)$ is the initial Hamiltonian, and apart from rescalings analogous to equations (5.50) and (5.51), $\mathscr{H}_\tau(\phi')$ is the effective Hamiltonian for a system in which fluctua-

tions with wavevectors $|q| > e^{-\tau}\Lambda$ ($\tau > 0$) are damped out, smoothly. This is achieved by choosing a function $\alpha_q(\tau)$ such that as τ increases $W(\phi', \phi, \alpha_q(\tau))$ approaches 1 for such fluctuations. A qualitative form for $\alpha_q(\tau)$ is shown in figure 13. The functional differential equations obeyed by $\mathscr{H}_\tau(\phi)$ are discussed in Wilson and Kogut (1974). The ϵ expansion is rather complicated in this formalism but can be handled at low order; critical exponents are independent of the precise form of cutoff (Golner and Riedel 1975a, Shukla and Green 1975, Rudnick 1975a, Bissett 1975). However, one may hope to find alternative approximation schemes (Golner and Reidel 1975b) in this formalism.

In the general formalism (5.67) one certainly does encounter redundant variables (see the discussion and references in §4.1).

(vi) In the approximate recursion formula of Wilson (1971a, b), Wilson and Kogut (1974, p117), one neglects the momentum dependence of all contributions to the effective Hamiltonian $\tilde{\mathscr{H}}(\sigma)$, defined in equation (5.44). $\tilde{\mathscr{H}}(\sigma)$ then becomes a power series in σ. This approximation is motivated by the observation that the scale factor ζ in (5.51) differs from that obtained by naive dimensional analysis

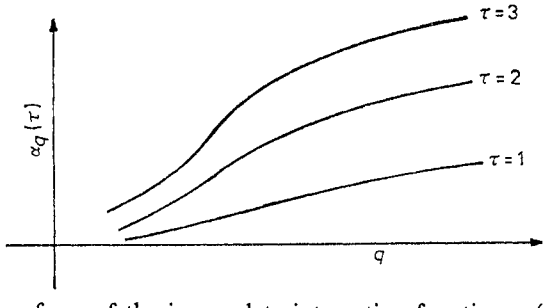

Figure 13. Qualitative form of the incomplete integration function $\alpha_q(\tau)$ in equation (5.67).

by terms of order ϵ^2. In the approximate recursion formula all contributions to $(\nabla\phi')^2$ are neglected, so that $\zeta = b^{(d-2)/2}$, and hence $\eta \equiv 0$ automatically. This is a good approximation to first order in ϵ expansion since $\eta = O(\epsilon^2)$ and a tolerable approximation in three dimensions where $\eta \simeq 0{\cdot}04$. Within this approximation the field variable $\phi(x)$ may be replaced by a single variable ϕ without loss of generality, and the recursion formula looks like

$$\exp\left(-Q'(\phi)/b^d\right) = \frac{\int_{-\infty}^{\infty} d\psi \exp\left[-\psi^2 - \tfrac{1}{2}Q(b^{1-d/2}\phi + \psi) - \tfrac{1}{2}Q(b^{1-d/2}\phi - \psi)\right]}{\int_{-\infty}^{\infty} d\psi \exp\left[-\psi^2 - \tfrac{1}{2}Q(\psi) - \tfrac{1}{2}Q(-\psi)\right]} \tag{5.68}$$

where $Q(\phi) - r\phi^2$ represents interactions of the form $u\phi^4 + w\phi^6 + \ldots$. This expression can be used to obtain critical exponents of all interactions of the form $\Sigma a_n \phi^n$ to first order in ϵ (Wegner 1972b). It can also be used to obtain numerical results in three dimensions (Wilson 1971a, b). Wilson has also studied the fixed point and exponents obtained from (5.68) in d dimensions (Wilson and Kogut 1974) and analysed the results to indicate the asymptotic nature (zero radius of convergence) of the ϵ expansion. (The convergence of the ϵ expansion will be considered in §6.1 when results are tabulated.) Golner (1973a) has derived a modified form of the recursion formula in which η is not necessarily zero.

These above remarks do not cover all the recursion formulae in the literature but should give an indication of their variety.

5.5. RG in renormalised perturbation theory

In the previous section we outlined the Wilson–Fisher recursion formula, and many developments of it. We now discuss qualitatively the theory of renormalisation, whose development led to the first RG formalism of Stueckelberg and Petermann (1953) and Gell-Mann and Low (1954).

The theory of renormalisation was developed in the early 1950s to handle the problem of ultraviolet divergences in relativistic quantum field theories (RQFT) such as quantum electrodynamics. Remarkably, the formalism of such theories of elementary particle interactions is very similar to that of the statistical mechanics of a field, with the main differences as follow.

(i) $\exp(-\mathscr{H})$ in (5.17) is replaced by $\exp(iA)$, where A is the action. For a theory describing interacting spin-0 Bose particles, a typical form of action is

$$A=\int \mathrm{d}^4x\{\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-(1/4!)u\phi^4\} \tag{5.69}$$

where $\int \mathrm{d}^4x$ is over four-dimensional space–time, and $\partial_\mu\phi\partial^\mu\phi$ is the relativistic invariant $(\partial_t\phi)^2-(\nabla\phi)^2$ in units in which the velocity of light is 1. Up to corrections induced by the self interactions, m^2 is the mass of the particle.

(ii) The correlation functions $\langle\phi(x_1)\ldots\phi(x_N)\rangle$ in (5.17) become the Green functions of the quantum field theory, from which scattering amplitudes are calculated.

The Feynman rules for calculating Green functions in RQFT as a power series in u are therefore extremely close to those for calculating correlation functions in statistical mechanics. However it is conventional in RQFT to presume that the form of interaction (5.69) is correct to arbitrarily high momentum, or short distances—there is no analogue of a lattice spacing in space–time. In this case the discussion in §5.3.1 indicates enormous problems, because if there is no cutoff ($\Lambda\to\infty$) the factors $\ln \Lambda/q$, $\ln \Lambda/m$ which accompany the expansion in u are infinite.

The problem is therefore that if one tries to calculate scattering amplitudes in terms of u and m, the parameters in the action (5.69), there are meaningless divergences. The renormalisation programme to overcome this difficulty is as follows. In order to obtain meaningful initial expressions one *regularises*, i.e. puts into the theory some kind of cutoff. Scattering amplitudes can therefore be calculated in terms of u, m and Λ. There are divergences in the limit $\Lambda\to\infty$, of course. Now, instead of writing the scattering amplitudes as functions of u and m, one writes them in terms of the physical or renormalised quantities u_R, m_R. For example, the physical coupling constant u_R could be defined as the value of the scattering amplitude of two particles→two particles at threshold (zero three-momentum $\boldsymbol{q}$). One now expresses scattering amplitudes in terms of u_R and m_R, and Λ, instead of u and m. (Technically, a change in the scale of Green function, analogous to the rescaling (5.51), is also required.)

The 2→2 scattering amplitude written in this way has one obvious property: at threshold it equals u_R, and hence is *finite* in the limit $\Lambda\to\infty$, with u_R and m_R fixed. The theory of renormalisation is designed to discover the types of theories for which *all* scattering amplitudes are finite for *all* values of momenta in the limit $\Lambda\to\infty$ (at fixed physical couplings and masses). The class of such *renormalisable* theories is non-empty, and includes (5.69), quantum electrodynamics and non-Abelian gauge theories.

This theory of renormalisation in RQFT has been enormously developed during

the past 25 years. We list only some general references: Bogoliubov and Shirkov (1959), Bjorken and Drell (1965), Lowenstein (1972), Zimmermann (1971) and Taylor (1976). A recent review is given by Delbourgo (1976).

The RG in this formalism arises through the recognition that there is an arbitrariness in the reparametrisation of the scattering amplitudes. There is no need to rewrite them in terms of the value u_R at threshold—one may choose to use the value $u_R(\tau)$, at *any* momentum scale $e^{-\tau}\mu_0$ where μ_0 is some fixed momentum scale. The *same* theory can therefore be written in terms of any one of a continuous family $u_R(\tau)$ of coupling constants, which give the effective interaction strength at a momentum scale $e^{-\tau}\mu_0$. This family of equivalent coupling constants for any given theory is obtained as the solution of a differential equation (as in equations (5.60)–(5.66))

$$du_R(\tau)/d\tau = \beta(u_R(\tau))$$

with initial conditions $u_R(0) = u_R$, say. The function β can be calculated from the Feynman graph expansion as a power series in its argument.

Let us now indicate how this approach is applied to critical phenomena in statistical mechanics, where one has a cutoff Λ, and is interested in the region $q \ll \Lambda$, $T - T_c \ll \Lambda^2$, etc. The idea is to introduce an arbitrary momentum (wavevector) scale μ_0, such that

$$q,\ (T - T_c)^{1/2} \ll \mu_0 \ll \Lambda.$$

One then writes the correlation functions in terms of the renormalised (dimensionless) coupling u_R defined at the momentum scale μ_0. In the same spirit that the dimensionless parameter appearing in the Hamiltonian has some arbitrary value, we permit u_R to have any unexceptional value. Now we can make an approximation for the correlation functions based on the fact that $\mu_0/\Lambda \ll 1$, namely we can let $\Lambda \to \infty$, and neglect the corrections which are typically of the form $(\mu_0^2/\Lambda^2)^{1-m\epsilon}$ for some integer m, i.e. $(\mu_0^2/\Lambda^2)\ \ln^p(\mu_0/\Lambda)$ in ϵ expansion. The limit $\Lambda \to \infty$ in this case is not a necessity forced on us because there is no physical cutoff, as is envisaged in RQFT, but is simply an approximation which permits us to neglect small terms in correlation functions which are written in terms of u_R.

Finally we use the freedom of choice of the momentum scale to define $u_R(\tau)$, the effective coupling at the momentum scale $e^{-\tau}\mu_0$. In the limit $\tau \to \infty$, $u_R(\tau)$ gives the effective coupling at the infrared momentum scales in which we are interested. To describe this behaviour we look for fixed points u_R^* and study the behaviour close to fixed points as in equations (5.61)–(5.66).

Renormalised perturbation theory is an enormous subject. We restrict ourselves to the following remarks.

(i) The scaling properties of correlation functions and free energy, the form of corrections to scaling, etc, are compactly described by a set of differential equations.

(ii) A particularly attractive feature is that the differential equation for the ϕ^4 coupling, say, does not depend on the coefficient of ϕ^6, etc, in contrast to the Wilson–Fisher approach discussed in §5.4. The effect of these so-called composite operators (ϕ^6, etc) can in principle be calculated; the critical exponents associated with them can be obtained as a power series in ϵ.

(iii) The precise form of these differential equations depends on the choice of renormalisation procedure. An indication of the variety of these can be found in Zinn-Justin (1973), 't Hooft (1973), Weinberg (1973), Lowenstein (1972), Schroer

(1973) and Mitter (1973). One obtains the same result for universal quantities (exponents, scaling functions, etc) independent of the particular renormalisation procedure.

Equivalence to the other kinds of RG discussed in §§4 and 5 is still not completely clarified, although there is always agreement in practice (see for example the discussion of the ϵ expansion in Kadanoff's variational method (Kadanoff *et al* 1976)). For a general discussion see Benettin *et al* (1977).

(iv) The asymptotic form of correlation functions (the limit $\xi^{-1} \ll q$ ($\ll \mu_0, \Lambda$)) is studied by the Callan–Symanzik equation (Callan 1970, Symanzik 1970) in conjunction with the operator product expansion (Kadanoff 1969, Wilson 1969, Symanzik 1971, Mitter 1973. See for example Brézin *et al* (1974a)). The relationship of the Callan–Symanzik equation to the RG is discussed in, for example, Benettin *et al* (1977).

(v) Renormalised perturbation theory is a particularly efficient tool for ϵ-expansion calculations (see the results tabulated in §6.1).

(vi) The above references are less than a smattering of the literature on renormalised perturbation theory. We recommend the interested reader to pursue the articles by Brézin *et al* (1976a) and Amit (1977).

6. Applications

In this final section we give some references to applications of the techniques discussed in the previous sections. Mostly we shall be concerned with applications to phase transitions and related problems (§§6.1–6.7); this is because the richest and most successful results have been obtained here. Those who are not specifically interested in phase transitions may regard these sections as an indication of what can be done. Section 6.8 contains an extremely brief review of applications in elementary particle physics. Although the results of the RG in elementary particle physics are not as conclusively confirmed in experiment, this section is heavily underweighted in view of the contribution to RG theory made in the study of relativistic quantum field theory. Section 6.9 contains a ragbag of other applications which are collected there mainly because of our incompetence to deal with them at any length in the separate sections they merit. Some concluding remarks are made in §6.10. Throughout this section the references listed are at best a ramified set from the literature.

6.1. The n-vector model—short-range interactions

6.1.1. The classical n-vector Heisenberg model. In stating results, it is convenient to generalise the Ising-like models considered up to now to Heisenberg-like models, where the basic degree of freedom is an n-component vector ($\phi_a(x)$ in a field model, $s_m{}^a$ in a lattice model with $s_m{}^2 = 1$ at each site m). The basic Hamiltonian is O(n)-invariant but otherwise has the same form as equations (2.3) or (5.14). The magnetic field H_a, or its analogue, is a vector and the external field term is, e.g. $H_a \Sigma_m s_m{}^a$.

The Ising model corresponds to $n = 1$, and should be applicable to uniaxial magnets, the liquid–gas system (through the lattice gas model—see Griffiths (1972), or more generally Hubbard and Schofield (1972a)) and binary alloy mixtures. The $n = 2$ case should be applicable to magnets with an easy plane of magnetisation, and to the superfluid transition in ^{4}He. The $n = 3$ case is applicable to isotropic three-

dimensional magnets. There are also many physical realisations of models with an order parameter which is an n-component vector with $n \geqslant 4$. In general these models have a lower symmetry than O(n). We shall discuss them later. (See Kadanoff *et al* (1976) and Fisher (1967a) for more details on the various types of critical systems.)

The O(n)-invariant model is almost as easily handled in field theory as the one-component model. The dependence on n of fixed point, critical exponents, etc, can be explicitly displayed in the ϵ expansion. In contrast there are problems in implementing the RG in the lattice n-vector model. They will be discussed in §6.5. Since RG results in Ising lattice models are fairly thoroughly reviewed in §4, we concentrate here on field theory results for the n-vector model in $4-\epsilon$ dimensions.

6.1.2. Principal critical exponents; convergence of ϵ expansion. Early references on ϵ-expansion calculations are given in §5.4. The results

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 + \frac{n+2}{8(n+8)^5}[n^4+44n^3+664n^2+2496n+3104-48(5n+22)\zeta(3)]\epsilon^3 + O(\epsilon^4) \tag{6.1}$$

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \frac{(n+2)}{8(n+8)^4}(-n^2+56n+272)\epsilon^3 + \frac{(n+2)}{32(n+8)^6}[-5n^4-230n^3+1124n^2+17920n+46144 - 384(5n+22)\zeta(3)]\epsilon^4 + O(\epsilon^5) \tag{6.2}$$

were obtained by Brézin *et al* (1973d). The factors of $(n+2)$ persist at all orders in perturbation theory and imply that the continuation to $n=-2$ has mean-field exponents (Balian and Toulouse 1973, Fisher 1973). The major critical exponents controlling leading singularities are obtained from these expressions by the scaling laws (3.67), (3.94) and (3.98). They are tabulated in, for example, Wilson and Kogut (1974). $\zeta(3)$ ($\simeq 1{\cdot}20$) is the Riemann ζ function.

The numerical contributions to γ and η for $n=1$, 2, 3 are given in table 2. The sums to ϵ^2 for γ (ϵ^3 for η) agree very well with the series results for $d=3$ ($\epsilon=1$). The agreement is distinctly poorer when the ϵ^3 terms (for γ; ϵ^4 for η) are added.

Table 2. Contributions to γ and η at low orders in ϵ compared with series results.

Exponent	n	ϵ^0	ϵ	ϵ^2	ϵ^3	ϵ^4	Sum ($\epsilon=1$)	Series
γ	1	1	0·167	0·077	0·040–0·074 $\zeta(3)$	—	1·195	1·250±0·005
	2	1	0·2	0·1	0·056–0·077 $\zeta(3)$	—	1·263	1·318±0·01
	3	1	0·227	0·119	0·069–0·076 $\zeta(3)$	—	1·324	1·405±0·02
η	1	0	0	0·019	0·019	0·011–0·017ζ (3)	0·029	0·047±0·01
	2	0	0	0·020	0·019	0·011–0·016ζ (3)	0·031	0·04±0·01
	3	0	0	0·021	0·018	0·009–0·014ζ (3)	0·031	0·04±0·01

The problem of convergence is even more pressing for the correction to scaling exponent ω (equation (5.65)):

$$\omega = -\epsilon + \frac{3(3n+14)}{(n+8)^2}\epsilon^2 - \frac{1}{(n+8)^3}\left(\frac{33}{4}n^2 + \frac{461}{2}n + 740 + 24(5n+22)\zeta(3)\right)\epsilon^3 + \frac{18(3n+14)^2}{(n+8)^4}\epsilon^3 + O(\epsilon^4) \tag{6.3}$$

(Brézin *et al* 1973b).

Attempts to improve convergence or resolve the problem have been made by Nickel (1974), Colot *et al* (1975) and Baker *et al* (1976). Nickel (1974) recognises the probable asymptotic nature (zero radius of convergence—see Wilson and Kogut (1974 p124) of the ϵ expansion and calculates a sensibly chosen subset of higher-order graphs. When the contributions of these graphs are summed by continued fractions, their net effect is to reduce the coefficient of the $\zeta(3)$ terms in (6.1) and (6.2) by a factor of $\sim 0{\cdot}4$. His final results are, for example:

$$\begin{aligned} n=1; &\qquad \gamma = 1{\cdot}250 \pm 0{\cdot}005 \\ n=1; &\qquad \eta = 0{\cdot}0413 \pm 0{\cdot}001 \end{aligned} \tag{6.4}$$

in excellent agreement with the series expansion results.

Nickel has also calculated the main RG functions to fifth non-trivial order in the ϕ^4 coupling in *three* dimensions using the Callan–Symanzik formalism. A typical result is ($n=1$)

$$\beta(v) = -v + v^2 - 0{\cdot}422v^3 + 0{\cdot}351v^4 - 0{\cdot}376v^5 + 0{\cdot}496v^6 + \ldots \tag{6.5}$$

where v is a suitably normalised coupling constant. There is now no small parameter ϵ to enable us to find a systematic zero of this function (to give u^*, see equation (5.62)) but there are enough terms to enable Padé–Borel resummation technique to be applied to find a zero numerically. This is reported in Baker *et al* (1976), who quote the results ($n=1$, $d=3$)

$$\begin{aligned} \gamma &= 1{\cdot}241 \pm 0{\cdot}002 \\ \eta &= 0{\cdot}02 \pm 0{\cdot}02 \\ \omega &= -0{\cdot}78 \pm 0{\cdot}01. \end{aligned} \tag{6.6}$$

(The conventional definition of ω, and sometimes of $\beta(u)$, has an opposite sign.)

The most remarkable developments on the problem of divergence of the Feynman graph expansion and the ϵ expansion have been made recently. Following a new steepest descent expansion using instantons, Lipatov (1976) and Brézin *et al* (1977a, b) have shown how to obtain systematically the behaviour at asymptotic orders in perturbation theory. The idea of controlling asymptotic orders first appeared in the anharmonic oscillator (Bender and Wu 1973, 1976; see also Langer 1967, Parisi 1977). We quote the results of Brézin *et al* (1977a, b): the series are of the form

$$\sum_K \epsilon^K K!(-1)^K a^K K^b c(1 + O(K^{-1}))$$

with

$$a = \frac{3}{n+8} \qquad b = \begin{cases} 3 + \frac{1}{2}n & \text{for } \eta \\ 5 + \frac{1}{2}n & \text{for } \omega. \end{cases} \tag{6.7}$$

These developments provide a very rich field for exploration. The divergence problem has not been studied to the same order or depth for other physical quantities; we shall adopt the philosophy that the low orders in ϵ expansion do appear to give a reasonable quantitative picture.

6.1.3. Free energy and equation of state. The scaling function $h((T-T_c)/M^{1/\beta})$, equation (3.68), is known to order ϵ^2 for general n and ϵ^3 for $n=1$ (Brézin *et al* 1972, 1973e, Avdeeva and Migdal 1972, Avdeeva 1973, Wallace and Zia 1974). Convergence towards the series results of Gaunt and Domb for the Ising model are shown in figure 14. Convergence is not as good for general n. (Here and henceforth we direct the readers to the references for explicit results.)

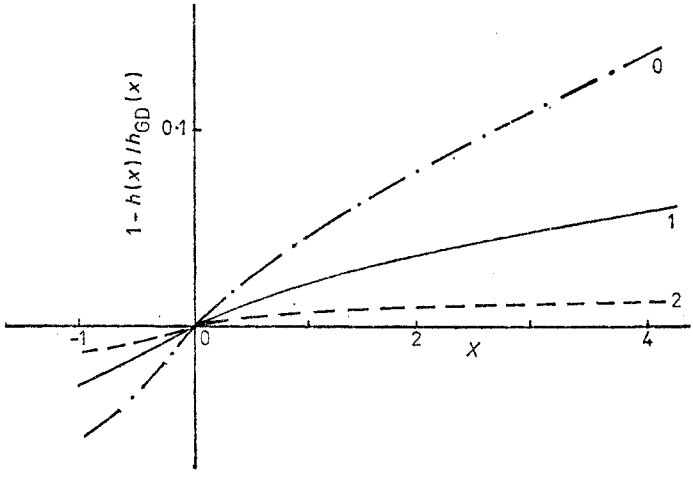

Figure 14. Comparison of the $n=1$ equation of state function $h(x)$ at order 0, 1 and 2 in ϵ, with the numerical results of Gaunt and Domb (1970) (from Brézin *et al* 1973e).

There are many interesting properties of $h(x)$.

(*a*) It is in general impossible to write a closed explicit form for $h(x)$ which obeys all of the analyticity requirements of Griffiths (1967). A particularly elegant way of preserving scaling and analyticity requirements is to write the equation of state in a parametric form (Josephson 1969, Schofield 1969, Schofield *et al* 1969). It turns out that the simplest such model, the linear parametric model, is an exact representation of the Ising equation of state to order ϵ^2 (Brézin *et al* 1972). It must be modified to incorporate the ϵ^3 results (Wallace and Zia 1974). It is not satisfactory for $n \neq 1$ even at order ϵ.

(*b*) The region of spontaneous symmetry breaking (the coexistence curve $T < T_c$, $H=0$) is particularly interesting. For $n \neq 1$, there are Goldstone modes because of the spontaneous breaking of a continuous symmetry. These are the spin waves of a dynamical treatment, and in equilibrium statistical mechanics are manifested by infinite transverse susceptibility. Given the equation of state

$$H_a = M_a |M|^{\delta-1} h((T-T_c)/|M|^{1/\beta})$$

the inverse susceptibility tensor is

$$\frac{\partial H_a}{\partial M_b} = \delta_{ab}|M|^{\delta-1}h((T-T_c)/|M|^{1/\beta}) + M_a M_b |M|^{\delta-3}\tilde{h}((T-T_c)/|M|^{1/\beta}).$$

For changes in magnetisation perpendicular to H, this tensor has $(n-1)$ eigenvalues:

$$(\chi_\perp)^{-1} = |M|^{\delta-1} h((T-T_c)/|M|^{1/\beta})$$

so that

$$\chi_\perp = M/H. \tag{6.8}$$

Clearly $\chi_\perp \to \infty$ as $H \to 0$ ($T < T_c$, i.e. $M \neq 0$).

Therefore, for a system with continuous symmetry, every point along the entire coexistence curve ($H=0$, $T<T_c$) is a critical point in the sense that the transverse susceptibilities (6.8) are infinite there. It turns out that these critical transverse fluctuations do produce additional singularities for all $T<T_c$. They induce a divergence of the usual longitudinal susceptibility:

$$\chi_\parallel \equiv \frac{\partial M}{\partial H_\parallel} \sim aH^{-\epsilon/2} + bH^0 + O(1) \qquad (H \to 0,\ T < T_c \text{ fixed}) \tag{6.9}$$

where $\epsilon = 4-d$ as usual. This behaviour is believed to be exact for all $\epsilon (0 < \epsilon < 2)$. The coefficient a has a factor $(n-1)$ so that the singularity $H^{-\epsilon/2}$ is absent for the Ising system. The exponent γ is therefore not defined for a Heisenberg system ($n \neq 1$) below T_c. The form (6.9) first appeared in spin-wave theory (Holstein and Primakoff 1940). The ϵ expansion of the singularity (6.9) also appears in the ϵ-expansion calculations. The scaling generalisation of (6.9) when T is close to T_c can be resurrected from the equation of state function $h(x)$ (Brézin *et al* 1973e, Wallace and Zia 1975b). An incomplete (in our view) attempt to describe the entire critical region including the coexistence curve has been given by Nelson (1976). Further references and discussion can be found in Wallace (1976).

(*c*) There are also subtleties about the coexistence curve in Ising-like models, because of the existence of fluctuations corresponding to configurations in which large regions of the *different* phases coexist. Such fluctuations are not included in conventional graph calculations which deal with fluctuations in a single ordered phase; they correspond more to soliton-like objects. The cluster or droplet model of condensation (see Langer (1967) and Fisher (1967b) for references) does attempt to take these into account and predicts infinitely differentiable (in H) singularities at $H=0$. A lattice RG approach to this problem which reproduces many aspects of the droplet model is discussed by Klein *et al* (1976) (see also Subbarao 1976b).

The specific heat and direct calculations of the free energy are discussed by Brézin *et al* (1976a).

6.1.4. Correlation functions. The two-spin correlation function has been calculated by several methods in ϵ expansion. Fisher and Aharony (1973, 1974) have determined the scaling function A in equation (3.96) for zero external field and above T_c by direct exponentiation of logarithms in the Feynman graph method (Wilson 1972). Only the ϵ expansion of the asymptotic form (3.104) is obtained in this way, but one may check consistency of the logarithms with the form (3.104). The coefficients may then be reconstructed. Hubbard (1974) and Combescot *et al* (1974, 1975) extend this calculation to $H \neq 0$ and $T < T_c$. A proper treatment of the leading and next-to-leading terms for $q \gg \xi^{-1}$ was given by Brézin *et al* (1974a, b) using the Callan–Symanzik equation and the operator product expansion. Some universal critical amplitudes associated with correlation functions are reported in Brézin *et al* (1974c) and Aharony (1974a). More detailed discussions in the RG formalism are

given by Bervillier (1976). Reviews of scaling in correlation functions are given by Brézin *et al* (1976a), Fisher (1974a) and Fisher and Jasnow (1977).

It is worth remarking that particularly accurate representations of the two-spin correlation function can be obtained by combining the scaling property with a dispersion relation approach using the pole and cut structure along the imaginary momentum axis (Ferrell and Scalapino 1975, Bray 1976).

6.1.5. Corrections to scaling, anisotropy and crossover. This is an enormous subject which could be given an extensive coverage. We restrict ourselves to the following remarks.

The first question concerns the stability of the Heisenberg fixed point to perturbations by other interactions. If we take an interaction, symbolically, $u_{MN}\int \mathrm{d}^d x \mathrm{O}(\nabla^M \phi^N)$ then the coupling constant has the Λ dependence:

$$u_{MN} = g_{MN}\Lambda^{d-M-N(d-2)/2}$$

according to the dimensional analysis equation (5.32); g_{MN} is dimensionless. The discussions of §5.3 suggest that a perturbation in u_{MN} will be important if

$$d - M - N(d-2)/2 \geqslant 0$$

and unimportant otherwise. This naive analysis is modified by the ϕ^4 interactions which build up powers of Λ^{4-d} at each order in ϕ^4. A proper RG calculation which enumerates the scaling variables and calculates their critical exponents is required.

To this end one notes that the RG transformation has a change of scale as the main ingredient. It commutes with $\mathrm{O}(d)$ and $\mathrm{O}(n)$ rotations in the coordinate or field spaces respectively. Hence a decomposition of all possible interactions into irreducible representations of the symmetry group of the fixed-point Hamiltonian ($\mathrm{O}(d)\times\mathrm{O}(n)$ in this case) produces a partial factorisation of the problem of finding scaling variables; only interactions belonging to the *same* irreducible representation of $\mathrm{O}(d)\times\mathrm{O}(n)$ will be mixed.

This result is explicit in Wegner (1972a) where the approximate recursion formula (Wilson 1971a, b) is used to derive exponents of scaling variables to order ϵ. The scaling variables can be written as harmonic ($\mathrm{O}(n)$) polynomials of degree k multiplied by polynomials in ϕ^2 specified by k and the equivalent of a 'principal quantum number' m. It is also explicit in Brézin *et al* (1974d) who study an arbitrary interaction $u_{abcd}\phi_a\phi_b\phi_c\phi_d$. The $\mathrm{O}(\phi^4)$ scaling variables of the Heisenberg fixed point are the three classes of interactions $(\phi^2)^2$, $\phi^2\phi_a\phi_b u_{ab}$, $\phi_a\phi_b\phi_c\phi_d u_{abcd}$ where $u_{aa}=0=u_{aacd}$. These correspond to $k=0$, 2, 4 in Wegner's notation. The critical exponents have been obtained to rather high order in ϵ for interactions which by their transformation properties under $\mathrm{O}(d)\times\mathrm{O}(n)$ cannot mix with other interactions and are hence scaling fields (Wilson and Kogut 1974 p138, Ketley and Wallace 1973, Wallace and Zia 1975a). Numerical investigation of the $n=3$ model using the approximate recursion formula in three dimensions is reported by Grover *et al* (1972).

The implications of these calculations for three-dimensional systems are as follows. Interactions involving derivatives of fields or powers of the field $\mathrm{O}(\phi^N)$, $N\geqslant 5$ are irrelevant for the Heisenberg fixed point in $d=3$. The critical exponents governing leading singularities are unaffected by such terms. Terms of order ϕ^N, $N\leqslant 3$ certainly are relevant. The coefficients of such terms must be controlled to be 'small' if Heisenberg critical behaviour is to be observed. If the system is intrinsically

O(n)-invariant there are only two relevant variables, H and $(T-T_c)$ corresponding to interactions ϕ_a, ϕ^2; if H and $T-T_c$ are small one obtains the scaling equation of state discussed previously. If the system is not O(n)-invariant, relevant interactions $u_{ab}\phi_a\phi_b$ ($u_{aa}=0$) and $u_{abc}\phi_a\phi_b\phi_c$ may be present. For the O(ϕ^4) interactions we know that $(\phi^2)^2$ is irrelevant; it turns out that the $k=2$ interaction $u_{ab}\phi_a\phi_b\phi^2(u_{aa}=0)$ is also irrelevant. The $k=4$ interaction $u_{abcd}\phi_a\phi_b\phi_c\phi_d$ ($u_{aacd}=0$) is irrelevant (relevant) if n is less (greater) than a critical number, n_c, which depends on d. In the ϵ expansion, $n_c=4-2\epsilon+O(\epsilon^2)$. Nickel (1974) suggests that $n_c\simeq 3$ for $d=3$. Irrelevant anisotropic interactions will affect the transverse susceptibility for $T>T_c$, $H\to 0$ in otherwise isotropic systems; they are 'dangerous irrelevant variables'.

For relevant interactions one must go beyond linear perturbations about the Heisenberg fixed point, and consider the crossover scaling function, of the form (3.56) and (3.57) for the appropriate number of relevant variables. The effect of a relevant interaction may be crossover to:

(i) A smeared behaviour with no singularities (e.g. with an external field H);

(ii) A second-order transition governed by a different fixed point, with different critical exponents (an anisotropic interaction $u_{ab}\phi_a\phi_b$ will produce crossover to the O(m) Heisenberg fixed point if it makes m modes become critical before the remaining $n-m$ as T is lowered to T_c (Fisher and Pfeuty 1972));

(iii) A first-order transition (most ϕ^3 interactions and many relevant ϕ^4 interactions).

A general review of multicomponent anisotropic systems is given by Aharony (1976b). A phenomenological discussion is given by Riedel and Wegner (1969); see also Fisher (1968). Development of the theory is contained in Pfeuty *et al* (1974) and Riedel and Wegner (1974). Crossover behaviour due to spin anisotropy ($u_{ab}\phi_a\phi_b$) is discussed by Nelson and Domany (1976) (see also Nelson and Rudnick 1975), Horner (1976) and Kosterlitz (1976a) to order ϵ and Bruce (1975) and Bruce and Wallace (1976) to order ϵ^2. Further discussion of the effect of ϕ^3 interactions is deferred to §6.3.2. There are many physical realisations of multicomponent field models with several types of ϕ^4 coupling. Within the ϵ expansion and for $n>0$ one cannot obtain limit cycles or other recurrent behaviour, and critical exponents are real for any fixed point (Wallace and Zia 1975c). The non-availability of a fixed point in the space of ϕ^4 interactions is interpreted as implying the existence of a first-order phase transition (see e.g. Brézin *et al* 1974d). Papers studying these types of models are Cowley and Bruce (1973), Aharony (1973c, 1974d), Wallace (1973), Yamazaki (1976b), Nattermann and Trimper (1975) (cubic point group symmetry); Fisher and Nelson (1974), Aharony and Bruce (1974a), Nelson *et al* (1974), Bruce and Aharony (1975), Aharony (1975), Fisher (1975), Domany *et al* (1977), Domany and Fisher (1977) (bicritical and tetracritical points); Jones *et al* (1976) and Bailin *et al* (1977) (superfluid ^{3}He). Physical realisations of $n\geq 4$ vector models in structural transitions are discussed by Mukamel (1975), Mukamel and Krinsky (1975, 1976a, b), Mukamel *et al* (1976) and Bak and Mukamel (1976). Coupling of a non-critical field to critical fluctuations is discussed by Achiam and Imry (1975) and Achiam (1977).

6.2. 1/n expansion

Another advantage of considering the general O(n) model is that it is exactly soluble in the limit $n\to\infty$. This was established by Stanley (1968) who showed the equivalence of the classical Heisenberg lattice model in the $n\to\infty$ limit with the

exactly soluble spherical model (Berlin and Kac 1952, Langer 1965). The critical exponents are, for example:

$$\beta=\tfrac{1}{2} \qquad \gamma=\frac{2}{2-\epsilon} \qquad \eta=0 \tag{6.10}$$

and the scaling equation of state is

$$H/M^{\delta}=(t/M^{1/\beta}+1)^{\gamma}. \tag{6.11}$$

The development of a systematic $1/n$ expansion occurred simultaneously in lattice models (Abe 1972) and field models where Wilson recognised that summable subsets of graphs dominated in the limit $n\to\infty$ and at each order in $1/n$. The initial approaches produced logarithmic singularities at $1/n$, e.g.

$$\chi\propto(T-T_c)^{-2/(2-\epsilon)}\left[1+\frac{3\epsilon}{n}\frac{\sin(\pi\epsilon/2)}{(\pi\epsilon/2)}\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\epsilon/2)}\ln(T-T_c)+O\left(\frac{1}{n^2}\right)\right].$$

The logarithm then has to be interpreted as the $1/n$ expansion of the scaling behaviour $\chi\propto(T-T_c)^{-\gamma}$ with

$$\gamma=\frac{2}{2-\epsilon}-\frac{3\epsilon}{n}\frac{\sin(\pi\epsilon/2)}{(\pi\epsilon/2)}\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\epsilon/2)}+O(n^{-2}). \tag{6.12}$$

Here Γ is the usual Euler gamma function.

We remark:

(i) Identical results are obtained in $1/n$ expansion in lattice and field models—an explicit illustration of universality.

(ii) The $1/n$ expansion is complementary to the ϵ expansion, being valid for all $\epsilon(0<\epsilon<2)$; the ϵ expansion of (6.12) agrees with the $1/n$ expansion of (6.1). However, the $1/n$ expansion is poor numerically for $n\sim 3$, as may be expected from the factors, e.g. $(n+8)$ in (6.1).

(iii) A proper RG treatment of the $1/n$ expansion may also be given, eliminating the need for the assumption of exponentiation of logarithms.

These remarks hold for all calculations in $1/n$ expansion.

The literature on $1/n$ is also very extensive. (For reviews see Ma (1973b, 1976a) and Brézin *et al* (1976a, chap X).) Applications to topics covered in §6.1 include Ma (1972, 1973a), Ferrell and Scalapino (1972), Parisi and Peliti (1972), Abe (1972, 1973a), Bervillier *et al* (1974) (critical indices in the Heisenberg model, to order $1/n$, d arbitrary), Abe (1973b) (η to order n^{-2}, $d=3$), Brézin and Wallace (1973) (equation of state), Abe and Hikami (1973a, 1974a) (anomalous behaviour of specific heat amplitudes), Aharony (1973a, 1974c), Abe and Hikami (1973b) (correlation functions), Wallace (1973), Aharony (1974d), Hikami (1974), Bailin *et al* (1977) (effects of anisotropy), Balian (1975) (analyticity of critical temperature as a function of d), Ferrell and Scalapino (1974) and Bray (1974) (tests of convergence—the $1/n$ expansion is certainly also asymptotic).

6.3. The $n\to 0$ limit

It has been suggested that many physical properties can be formulated as the limit $n\to 0$ of various n-component models.

6.3.1. Polymers; the $n=0$ Heisenberg model. It was shown by de Gennes (1972b) that properties of repulsive chains of single polymers (i.e. in dilute solution) can

be obtained from the critical behaviour of the $n=0$ Heisenberg model. For example, the exponent ν governs the growth of the mean square size of a single polymer, proportional to $M^{2\nu}$ as the molecular weight $M\to\infty$. Values for exponents can therefore be read off from ϵ-expansion calculations. RG calculations on a lattice are given by Hilhorst (1977). Other physical properties of interest include the cross-over behaviour near the Θ temperature, where the effective repulsive interaction vanishes and we have typical random walk properties with $\nu=\frac{1}{2}$ and vanishing second virial coefficient. There are several field theory formulations on the market to describe these effects due to multipolymer interactions; their equivalence is not clear to us. Recent papers include Burch and Moore (1976a, b), des Cloizeaux (1974, 1975), Lawrie (1976) and Schäfer and Witten (1977). Mackenzie (1976) reviews scaling in polymers.

6.3.2. Random systems and expansions in $6-\epsilon$ dimensions. A trick due to Edwards and Anderson (1975) relates the calculation of the mean free energy of a quenched random system to the $n\to 0$ limit of an average on an unquenched system composed of n copies of the original:

$$\overline{\ln Z}=\lim_{n\to 0}\frac{1}{n}(\overline{Z^n}-1)$$

where Z is the partition function obtained by the thermal average and the bar indicates the average over the random interactions (see also Emery 1975, Grinstein and Luther 1976).

There are many different kinds of randomness; its qualitative features determine its effect. The phase transition may be smeared (e.g. McCoy and Wu 1968, 1969) or if it remains sharp, the transition temperature will in general drop as the random impurity concentration increases. Typical results are that critical behaviour in a second-order transition in a pure system is unchanged by dilute random impurities if the specific heat exponent α is negative and shows new critical behaviour controlled by a new fixed point if α is positive. An interesting new feature for the random Ising system is the appearance of a $\sqrt{\epsilon}$ expansion. Principal papers on this problem are Harris *et al* (1973), Harris and Lubensky (1974), Lubensky (1975), Grinstein and Luther (1976), Aharony (1976a) and Bak (1976).

Imry and Ma (1975), Lacour-Gayet and Toulouse (1974), Grinstein (1976) and Aharony *et al* (1976) study the case of a random external field. The problems of spin glasses and percolation can also be formulated as $n\to 0$ limits. The new feature here is that invariants *tri-linear* in the order parameter are permitted. The coupling constant of such terms has dimension $\Lambda^{(6-d)/2}$ (see equation (5.32)) and so we expect to find mean-field theory for $d>6$. It turns out that these models have second-order transitions describable by ϵ expansions in $6-\epsilon$ dimensions; the physical value is $\epsilon=3$, so convergence is a problem but results, particularly for the percolation problem, are rather encouraging. Principal references are Kasteleyn and Fortuin (1972), Harris *et al* (1975), Young and Stinchcombe (1975), Priest and Lubensky (1976), Kirkpatrick (1976), Dasgupta (1976), Amit (1976), Amit *et al* (1977) (percolation problem); Sherrington and Kirkpatrick (1975, 1977) and Harris *et al* (1976) (spin glasses and validity of $n\to 0$ limit).

Models with trilinear couplings have been suggested for physical systems, other than the above $n\to 0$ limits. The n-component Potts model (whose $n\to 0$ limit describes the percolation problem above) has been proposed for some magnetic and

displacive transitions (Weger and Goldberg 1973, Mukamel *et al* 1976, Aharony *et al* 1977), and a tensor model has been proposed for the isotropic to nematic transition in liquid crystals (de Gennes 1969, 1972a). Although ϵ expansions in $6-\epsilon$ dimensions can exist for these models (Priest and Lubensky 1976, Amit 1976) there is evidence that the ϕ^3 fixed points are unstable to ϕ^4 perturbations (Wallace 1977) and that these models have first-order phase transitions in three dimensions (Golner 1973b, Aharony *et al* 1977, Amit and Shcherbakov 1974, Rudnick 1975b, Zia and Wallace 1975b). However, note Baxter (1973) for exact results in two dimensions.

6.4. *Tricritical behaviour and ϕ^n ($n>4$) field theories*

If a thermodynamic system has a third parameter (in addition to $T-T_c$ and an external field) which can be used to vary the size of the ϕ^4 coupling, then a tricritical point may exist in which this effective coupling vanishes. At this tricritical point, the system is dominated by ϕ^6 interactions and hence according to equation (5.32) and the subsequent discussion mean-field theory holds for $d>3$ and in $d=3$ is altered only by logarithmic corrections. ^{3}He and ^{4}He mixtures, and polymers at the Θ temperature (see §6.3.1), are examples of such systems. The crossover from critical behaviour controlled by the Heisenberg fixed point to tricritical behaviour controlled by the Gaussian (free-field) fixed point is discussed by Riedel and Wegner (1969), Griffiths (1973) and Schulman (1973) (phenomenologically). The RG approach is outlined by Riedel and Wegner (1972) and discussed by Nelson and Rudnick (1975), Lawrie (1976), Nicoll *et al* (1976) in ϵ expansion ($\epsilon=4-d$), and by Amit and De Dominicis (1973) in $1/n$ expansion (see also Brézin *et al* 1976a). Exponents can be calculated (at lowest order in the appropriate ϵ) for multicritical fixed points controlled by ϕ^{2N} interactions in $d=2N/(N-1)-\epsilon$ dimensions (Stephen and McCauley 1973, Tuthill *et al* 1975).

Tricritical behaviour has also been studied extensively using the RG in lattice models (see for example Krinsky and Furman (1974), Burkhardt and Knops (1977) and references therein).

6.5. *Systematic RG expansions at low temperature*

It is known that Ising and Heisenberg models have a critical dimension d_c below which they have no phase transition at any finite temperature; $d_c=1$ for Ising models and 2 for n-component Heisenberg models (short-range interactions) (see Widom 1973, Mermin and Wagner 1960, Hohenberg 1967, Coleman 1973). One may expect that these models have T_c small in $1+\epsilon$ and $2+\epsilon$ dimensions respectively. If systematic low-temperature approximations can be made in RG calculations one may hope to obtain another ϵ expansion to complement the usual ones in $4-\epsilon$ dimensions.

Approximate lattice calculations were used by Migdal (1975b) to set up lowest-order ϵ expansions for Ising and Heisenberg ($n\geqslant 3$) in $d_c+\epsilon$ dimensions. The approximations give reasonable numerical results also in fixed dimension. However, systematic higher orders in ϵ are not yet obtainable in this approach. A field theory approach for $n\geqslant 3$ Heisenberg models due to Polyakov (1975) was developed by Brézin and Zinn-Justin (1976b, c) and Brézin *et al* (1976b, c) to make systematic calculations possible. Principal critical exponents corrections to scaling, anisotropic effects (see also Pelcovitz and Nelson 1976), equation of state, etc, are all calculable.

The $1/n$ expansion is also revealing (see also Hikami 1976). The basic idea of the calculations is to exploit the existence of $(n-1)$ Goldstone modes whose critical fluctuations dominate coexistence curve behaviour (see §6.1.3). One therefore studies models obtained from the usual Heisenberg model field theory by eliminating the longitudinal field by the constraint $\boldsymbol{\phi}^2 = \text{constant}$. This yields the static analogue of a spin-wave theory.

The case $n=2$ is special. It cannot exhibit spontaneous magnetisation in two dimensions (Mermin and Wagner 1960, etc, above) but does have a finite phase transition temperature at which singularities develop (see Berezinskii 1971, 1972, Kosterlitz and Thouless 1973, Kosterlitz 1974). The effects are probably too subtle to be reliably reproduced by the RG methods of §4, as attempted by Lublin (1975). A recent attack was made by José *et al* (1977).

6.6. Long-range interactions

So far we have considered only short-range interactions. It turns out that long-range interactions are *relevant* perturbations of the short-range Heisenberg fixed point and therefore give rise to new critical behaviour. The most significant result to date in this field was remarkably the first one, by Larkin and Khmel'nitskii (1969) who showed that uniaxial systems dominated by dipolar interactions would show mean-field behaviour modulated by logarithmic terms. These logarithmic features have subsequently been observed experimentally in $LiTbF_4$ by Ahlers *et al* (1975); see also Als-Nielsen (1976).

6.6.1. $1/R^{d+\sigma}$ interactions. If the exchange coupling in the lattice model falls off like $1/R^{d+\sigma}$ for large spin separation R, this gives rise to a term $\int_k k^\sigma s(k)s(-k)$ in the field theory Hamiltonian.

Naive dimensional analysis indicates this will dominate over the usual k^2 term if $\sigma<2$, and RG analysis shows it is relevant if $\sigma<2-\eta$. Given that the k^σ term then becomes dominant, further analysis shows that the ϕ^4 coupling has now dimension $\Lambda^{d-2\sigma}$ so that mean-field behaviour holds for $d>2\sigma$ and an ϵ expansion can be set up in $2\sigma-\epsilon$ dimension. Calculations paralleling those of §§6.1–6.5 can now be performed (see Aharony (1976b) for a review). Principal references are Fisher *et al* (1972), Sak (1973), Suzuki *et al* (1972), Suzuki (1972, 1973a, b, c) and Yamazaki (1976a). Exchange interactions which fall off like R^{-x} $(x<2)$ produce a finite T_c in one dimension; Kosterlitz (1976b) sets up an expansion with T_c of order $(2-x)$, in one dimension.

6.6.2. Dipolar interactions. Dipolar interactions fall off like $1/R^d$ but the angular dependence of the interactions distinguishes them from the case $\sigma=0$ of the above. They are present in ferroelectric and magnetic transitions. The dipolar interaction has much lower strength than the short-range exchange interaction in magnets except for those with weak exchange, i.e. low critical temperature.

Dipolar antiferromagnets are predicted to have the same critical behaviour as in the short-range case apart from some weak anisotropy effects. Dipolar ferromagnets are described by a strong dipolar fixed point; no matter how small the initial dipolar interaction, its effect becomes dominant towards the critical point. The size of the dipolar-dominated critical region is very small except for magnets with a very low T_c. Values of exponents at the dipolar-dominated fixed point are

numerically very similar to the short-range exponents, but crossover effects may be observable (for a review see Aharony 1976b). Main references are Aharony and Fisher (1973), Aharony (1973b), Bruce and Aharony (1974), Bruce (1974, 1976), Aharony and Bruce (1974b), Bruce *et al* (1976), and Nattermann and Trimper (1972, 1976).

The uniaxial dipolar interaction is a special case as mentioned earlier. References subsequent to Larkin and Khmel'nitskii (1969) are Brézin (1975a, b), Bervillier (1975), Nattermann (1975) and Brézin and Zinn-Justin (1976a).

6.6.3. Other long-range interactions. Aharony (1974b) has studied quadrupolar interactions.

When the lattice is allowed to become elastic, and the spins in a magnetic system, for example, are coupled to phonons, then long-range effects also arise: see Aharony (1973d), Wegner (1974b), Sak (1974), Rudnick *et al* (1974), Imry (1974), and Dohm and Kortmann (1974).

It is predicted (Halperin *et al* 1974c) that the long-range electromagnetic interactions will produce a small first-order transition in the superconducting phase transition. This is the non-relativistic analogue of what is called the Coleman–Weinberg (1973) phenomenon in relativistic quantum field theory.

Long-range interactions are also effectively built into so-called hierarchical models, which can be exactly solved by the RG method (Dyson 1969, Baker 1972, Gallavotti and Knops 1975, Baker and Golner 1977).

6.7. Dynamical critical phenomena

The extent of this subject really demands an entire review article to itself. Dynamic properties typically enter when one performs inelastic scattering experiments using light beams or neutron beams. The quantity measured is the dynamical structure factor $\tilde{G}(q, w)$, the Fourier transform with respect to space and time of the time-dependent correlation function:

$$G(x, \tau) = \langle (\phi(x, \tau) - \langle \phi(x, \tau) \rangle)(\phi(0, 0) - \langle \phi(0, 0) \rangle) \rangle$$

$$\tilde{G}(q, \omega) = \int \mathrm{d}^d x \mathrm{d}\tau \exp\left[-\mathrm{i}(qx - \omega\tau) G(x, \tau)\right].$$

Here ϕ is the density (e.g. particle density, magnetisation density, etc) by which the beam is scattered; it is not necessarily a scalar quantity but we drop indices on the field for simplicity. An introduction to dynamic phenomena is given in Stanley (1971).

A phenomenological scaling theory can be set up for $\tilde{G}(q, \omega)$ close to the critical point (Ferrell *et al* 1967, 1968, Halperin and Hohenberg 1969). The additional ingredient is the assumption of an exponent z associated with the frequency variable ω, so that in the usual magnetic notation:

$$\tilde{G}(q, \omega, t, M) = b^{2-\eta} \tilde{G}(bq, b^z \omega, b^{1/\nu} t, b^{\beta/\nu} M)$$

i.e.

$$\tilde{G}(q, \omega, t, M) = q^{-2+\eta} \tilde{g}(\omega q^{-z}, t q^{-1/\nu}, M q^{-\beta/\nu}).$$

The exponent z therefore governs the characteristic frequency, and hence time scale τ_q of modes of wavevector q: $\tau_q \propto q^{-z}$ at the critical point. Evidence for the dynamic scaling hypothesis is reviewed in Stanley (1971).

An introduction to microscopic models for describing dynamic phenomena is given by Ma (1976b). The starting point is a Langevin-like kinetic equation for $\partial\phi(x, \tau)/\partial\tau$ (or its Fourier transform with respect to x). The three terms in this equation describe typically the coupling to other modes, a damping force and random noise. These equations are not time-reversal-invariant and can describe relaxation processes, etc. They are best thought of as semi-microscopic equations with the random noise being due to modes with very short time scale which are not explicitly included in the kinetic equations. The form of the damping must respect all conservation laws in the system.

The main aim of the RG is a successive elimination of the modes with longer and longer relaxation times. The problem can be studied by ϵ expansions using a modified Feynman graph approach. The RG now operates in a space of couplings which includes damping coefficients as well as the static couplings. A major feature of the results is that systems with the same static behaviour may have very different dynamic properties, depending on the existence of coupling to other modes, or of conservation laws.

For an introduction to the RG approach, we refer the reader to Ma (1976b). Original references using the RG are Halperin *et al* (1972, 1974a, b, 1976), Ma and Mazenko (1974, 1975), Suzuki and Igarashi (1973), Suzuki and Tanaka (1974), Yahata (1974a, b), Kondor and Szépfalusy (1974), Abe and Hikami (1974b), Abe (1974), Freedman and Mazenko (1975) and Kawasaki and Gunton (1976). The multicomponent Bose fluid is considered by Halperin (1975a). Field theoretic techniques are a powerful tool also in dynamics: see De Dominicis (1975), De Dominicis *et al* (1975), Brézin and De Dominicis (1975) and Bausch *et al* (1976). Other early RG references may be found in Halperin's (1975b) review talk. Very recent progress includes discussion of deviations from scaling (De Dominicis and Peliti 1977) and application of expansions in $2+\epsilon$ dimensions (De Dominicis *et al* 1977).

6.8. Applications in relativistic quantum field theory

In this section we aim only to outline the philosophies behind the use of the RG in elementary particle physics. We thereby totally under-represent the role which this subject has played in the development of the RG.

We have already remarked that relativistic quantum field theories (RQFT) in three space and one time dimension are mathematically analogous to classical statistical mechanics in four space dimensions (§5.5). In statistical mechanics we have been studying infrared properties of a theory defined by a meaningful Hamiltonian with some form of cutoff at high momentum. Since space and time are continuous and have no discrete lattice structure to our knowledge, it is conventional in elementary particle physics to assume that there is no cutoff and no meaningful 'bare' Hamiltonian. It is therefore unconventional to describe elementary particle physics as an infrared phenomenon, mathematically analogous to phase transitions; model calculations have been done (Wilson and Kogut 1974 p138, Wilson 1973, Shei and Yan 1974, McKane *et al* 1976) but all of them have defects in three space and one time dimensions.

The conventional approach is to set up the theory so that effective couplings when momenta are at the physical mass scale—say at threshold—are free parameters which can be fitted to experimental values. One then proceeds to ask how the effective

interactions change as one looks at high momenta (short distances). Different types of experiments can probe different high momentum regions, e.g. scattering at large energy and momentum transfer in electron nucleon, neutrino–nucleon and proton–proton, or more general hadron–hadron, collisions. In a naive approach, one attempts to make simple models in which masses m can be dropped in the limit momenta $q \gg m$. One thereby obtains power law behaviours q^{α}. If all mass scales disappear in the limit $q \gg m$, then the powers α are determined by naive dimensional analysis. This approach provides a good phenomenological description of large momentum transfer (deep inelastic) electron–nucleon and neutrino–nucleon scattering (see Llewellyn-Smith (1972) for an introduction). In a field theory this result would be analogous to obtaining mean-field exponents with all interactions negligible.

In order to reproduce this behaviour of zero effective interactions at short distances in a field theory, one needs the zero interaction fixed point of the RG to be attractive as one goes to high momenta. Such a theory is called asymptotically free. It has been shown that non-Abelian gauge theories (see Taylor (1976) for an introduction) can be asymptotically free (Politzer 1973, Gross and Wilczeck 1973, G 't Hooft unpublished) and that they are the only kind of field theories with this property in four dimensions (Coleman and Gross 1973). As with ϕ^4 theories in four dimensions additional logarithmic dependence is predicted on top of the scaling obtained by naive dimensional analysis. Observed violations of scaling may be due to other factors, however (see Politzer (1974) for a review).

The fact that effective couplings become weak at short distances in non-Abelian gauge theories has the converse that they become strong at large distances. Current research is attempting to show that these forces may be sufficiently strong to bind permanently particles which experience them. If this can be achieved then one may be able to realise a RQFT of elementary particles as bound states of quarks which cannot themselves be observed in isolation—quark confinement. In studying this problem it has been found fruitful to introduce lattice versions of non-Abelian gauge theories (Wilson 1974a), with physics to be obtained by taking a suitable limit of zero lattice spacing. Migdal (1975a, b) has shown how to set up approximate RG on a lattice for these models. The absence of a phase transition at any finite temperature (in the lattice theory) is required to produce quark confinement. This appears to be a property of non-Abelian theories in four dimensions, but fortunately not of the Abelian theory quantum electrodynamics (cf §6.5). A review is given by Kadanoff (1977).

Non-Abelian gauge theories may also provide the vehicle for unified theories of weak and electromagnetic interactions of elementary particles. RG calculations play an important role here also (see, for example, Georgi *et al* 1974).

Finally we remark that a totally different application of the RG has been made in the so-called Reggeon field theory of high-energy scattering (see Abarbanel *et al* (1975) for a review).

6.9. Remarks on other problems

Scaling properties in *turbulence* have been recognised for a very long time (see Kraichnan (1975) for a review). Attempts have been made to apply the RG to this problem (see Forster *et al* 1976, Frisch *et al* 1976).

The RG has been successfully applied to the Kondo problem—dilute magnetic ions in a conductor. For a review and further references see Wilson (1975).

The quantum properties of spins should not play an important role in phase transitions, except in the limit of zero temperature. In fact, the quantum Ising model in a transverse magnetic field at $T=0$ and d dimensions is equivalent to the classical Ising model in $d+1$ dimension, the transverse field in the former playing the role of the disordering field, the temperature in the latter. This was shown for $d=1$ by Pfeuty (1970) and Suzuki (1971). The generalisation to arbitrary d was conjectured by Pfeuty and Elliott (1971) and others. It is justified by the techniques of Stoeckly and Scalapino (1975).

Scaling behaviour at zero temperature is discussed by Baker and Bonner (1975) and RG transformations on quantum lattice systems by Young (1975) and Brower *et al* (1977). Niemeijer–van Leeuwen type of calculations on $d=2$ quantum systems have been used to estimate the standard $d=3$ critical exponents (Friedman 1976, Subbarao 1976c).

The electron gas in one dimension is of great current interest. We refer the reader to the reviews by Luther (1976) and Solyom (1976).

The existence of critical exponents depending on a continuous parameter in the Baxter model (Baxter 1972) is understood in the RG as due to a line of fixed points (see, for example, Kadanoff and Wegner 1971, van Leeuwen 1975). The Thirring model (see e.g. Wilson 1970) displays similar behaviour in two dimensions.

In a finite magnet, all spins within a correlation length of a surface will feel the effect of that surface. Generalised scaling properties have been introduced for semi-infinite systems. References are contained in Binder and Hohenberg (1972), Barber and Fisher (1973), Lubensky and Rubin (1975) and Bray and Moore (1977).

6.10. Conclusions

The breadth of applications of the renormalisation group should be clear from the previous sections. There is no doubt that the range of applications will continue to grow. Of course, conclusive experimental verification of many of the predictions may never be possible but it is fair to say that there is overwhelming evidence for the gross features predicted by the renormalisation group—leading critical exponents and scaling functions of Heisenberg-like models and the logarithmic singularities in uniaxial dipolars, for example.

Although the practical success of the renormalisation group is well established, there are several fundamental aspects which are still at best only partially understood. This is particularly true in lattice model calculations. For example, under what conditions is one guaranteed to find fixed points of the transformation rather than some more complicated behaviour and what are the convergence properties of the approximation schemes?

No doubt this subject will be a hunting ground for theorists for some years to come.

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