NAME:

PHYSICS 4456 — INTRODUCTION TO QUANTUM MECHANICS II
Spring 2005 — First midterm exam — February 17, 2005

Please note: The Virginia Tech *honor code* applies to this midterm. This is a *closed book* exam. Please do *not* communicate with your fellow students ! You may ask me for assistance if you find the wording of any of the problem unclear. Please turn in your *entire* work at the end of this test.

Good luck and keep calm !!!

1. *Hydrogen atom: conceptual questions.* (**10 points**)
   The energy eigenvalues of the hydrogen atom bound states are given by $E_n = -E_1/n^2$, $n = 1, 2, \ldots$ integer. The Bohr radius is $a_0 = \hbar^2/m_e e^2 = 0.53\,\mathring{A}$, and the Rydberg energy $E_1 = \hbar^2/2m_e a_0^2 = 13.6$ eV.
   (a) For given principal quantum number n, give the allowed values for the angular momentum quantum numbers ℓ and m.
   (b) What is the degeneracy of the nth level E_n, including spin ?
   (c) Why do the energy eigenvalues *not* depend on ℓ and m ?
   (d) What is the typical size of the He^+ ion in the ground state, and what is its ionization energy ?
   (e) An electron–positron bound state is called *positronium.* What are its typical size in the ground state, and its dissociation energy ?

2. *Matrix representations.* (**10 points**)
   (a) In the energy eigenstate basis, provide the matrix representation for the Hamiltonian of the one-dimensional harmonic oscillator.
   (b) For a particle with spin $s = 3/2$, write down the matrix representations of $\vec{S}^2$ and S_z in the eigenstate basis of these operators.
   (c) For a particle with spin $s = 1/2$, determine the matrix representations of S_x and S_y in the eigenstate basis of S_z.
   (Recall $S_\pm = S_x \pm iS_y$, and $S_+|\uparrow\rangle = 0$, $S_+|\downarrow\rangle = \hbar\,|\uparrow\rangle$).

This test sheet has **two** *pages.* **Please turn over !**

3. *Hydrogen atom: expectation values.* (**18 points**)
Consider the hydrogen atom ground state wave function
$\varphi_{100}(\vec{r}) = A\, e^{-r/a_0}$, with $r = |\vec{r}|$.
(a) Determine the constant A.
(b) What is the most likely value of r to be found in a measurement ?
(c) Compute the expectation values $\langle r \rangle_{100}$ and $\langle r^{-1} \rangle_{100}$.
(d) Find the ground state expectation value of the kinetic energy.
(e) Initially, let the atom be prepared in the superposition state
$\psi(\vec{r}, t=0) = \frac{1}{\sqrt{5}}\, [2\, \varphi_{100}(\vec{r}) - \varphi_{210}(\vec{r})]$. What is $\psi(\vec{r}, t > 0)$?
What are the probabilities of measuring $L_z = 0$ and $|\vec{L}| = 0$ in this state ? Calculate the atom's mean energy.

Useful integrals : $$\int_0^\infty x^k\, e^{-ax}\, dx = \frac{k!}{a^{k+1}} \ .$$

4. *Electron in a magnetic field and oscillator potential.* (**12 points**)
An electron is exposed to a time-independent homogeneous magnetic field $\vec{B} = B\, \vec{e}_z$, and to an electrostatic potential $V(\vec{r}) = -e\Phi(\vec{r}) = Kx^2/2$. The Hamiltonian thus reads

$$H = \frac{1}{2\, m_e} \left[\vec{p} + \frac{e}{c}\, \vec{A}(\vec{r}) \right]^2 + \frac{K}{2}\, x^2 \ .$$

(a) Choosing the gauge $A_x = 0 = A_z$, $A_y = Bx$, show that the linear momentum components p_y and p_z are conserved.
(b) Explain the separation ansatz $\varphi_{nk_yk_z}(x, y, z) = \varphi_n(x)\, e^{ik_y y + ik_z z}/2\pi$, and derive the stationary Schrödinger equation for $\varphi_n(x)$.
(c) Determine the energy eigenvalue spectrum for this system.

Good luck and keep calm !!!