NAME:

PHYSICS 4456 — INTRODUCTION TO QUANTUM MECHANICS II
Spring 2005 — Second midterm exam — March 24, 2005

Please note: The Virginia Tech *honor code* applies to this midterm. This is a *closed book* exam. Please do *not* communicate with your fellow students ! You may ask me for assistance if you find the wording of any of the problem unclear. Please turn in your *entire* work at the end of this test.

Good luck and keep calm !!!

1. *Angular momentum addition.* **(14 points)**
   Consider the addition of two angular momentum operators $\vec{J}_1$ and $\vec{J}_2$, with corresponding quantum numbers j_1, m_1 and j_2, m_2, resulting in the total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$, whose quantum numbers are labeled j and m.
   (a) Name the complete set of operators for which the basis states of the *uncoupled representation* are eigenstates.
   (b) How many basis states are in this *uncoupled representation* ?
   (c) In contrast, the basis states of the *coupled representation* are simultaneous eigenstates for which set of operators ?
   (d) What are the possible values of j in terms of j_1 and j_2, and which values may m take ? Also, how is m related to m_1 and m_2 ?

2. *Perturbation theory.* **(10 points)**
   Consider a one-dimensional Hamiltonian $H_0 = p^2/2m + V_0(x)$ with $V_0(-x) = V_0(x)$ symmetric. Assume *non-degenerate* eigenstates of H_0.
   (a) Show that for *any antisymmetric* perturbation $H' = V'$, with $V'(-x) = -V'(x)$, the *first-order* perturbation theory corrections to the energy eigenvalues must vanish: $E_n^{(1)} = 0$.
   (b) Will the *second-order* corrections $E_n^{(2)}$ in general be zero, too ?
   (c) Name the three relativistic correction terms that are responsible for the fine structure in the spectra of single-electron atoms / ions. (No equations or explanations required.)

This test sheet has **two** *pages.* **Please turn over !**

3. *Spin superposition state.* **(14 points)**
The spin-dependent part of an electron's (spin $s = 1/2$) Hamiltonian in a homogeneous magnetic field $\vec{B} = B\,\vec{e}_z$ reads $H = \mu_B B\,\sigma_z$, where $\mu_B = e\hbar/2m_e c$ is Bohr's magneton.
(a) Find the energy eigenvalues of this system.
(b) The system is initially prepared in the state $\xi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.
Show that ξ is an eigenstate of σ_x, and compute $\langle H\rangle_\xi$.
(c) What is the state $\xi(t)$ at $t > 0$? What are the probabilities of measuring the energies found in (a) ?
(d) Determine the expectation values $\langle\sigma_x(t)\rangle_\xi$ and $\langle\sigma_z(t)\rangle_\xi$.

Pauli's spin matrices: $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ , \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ .$

4. *Perturbed two-dimensional harmonic oscillator.* **(12 points)**
The Hamiltonian for a symmetric two-dimensional harmonic oscillator reads

$$H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right) = \hbar\omega\left(a^\dagger a + b^\dagger b + 1\right).$$

(a) Confirm that the degeneracy of the nth energy level of H_0 with eigenvalue $E_n^{(0)} = n\,\hbar\omega$ is $g_n = n$.
(b) Consider the perturbation Hamiltonian

$$H' = K'x\,y = \frac{\hbar K'}{2m\omega}\left(a + a^\dagger\right)\left(b + b^\dagger\right),$$

and apply *first-order perturbation theory* to compute the energy shifts $E_1^{(1)}$ and $E_2^{(1)}$ for the ground and first excited states. [Recall $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.]
(c) Which condition on K' must be fulfilled for perturbation theory to provide a sensible approximation ?

Good luck and keep calm !!!