NAME:

PHYSICS 4456 — INTRODUCTION TO QUANTUM MECHANICS II
Spring 2006 — First midterm exam — February 16, 2006

Please note: The Virginia Tech *honor code* applies to this midterm. This is a *closed book* exam. Please do *not* communicate with your fellow students ! You may ask me for assistance if you find the wording of any of the problem unclear. Please turn in your *entire* work at the end of this test.

Good luck and keep calm !!!

1. *Hydrogen atom: conceptual questions.* (**9 points**)
   The energy eigenvalues of the (ideal) hydrogen atom bound states are given by $E_n = -E_1/n^2$, with $n = 1, 2, \ldots$. The Rydberg energy is $E_1 = \hbar^2/2m_e a_0^2 = 13.6\,\text{eV}$, and the Bohr radius $a_0 = \hbar^2/m_e e^2 = 0.53\,\mathring{A}$.
   (a) For given principal quantum number n, give the allowed values for the angular momentum quantum numbers ℓ and m.
   (b) What is the degeneracy of the nth level E_n (neglecting spin) ?
   (c) Why do the energy eigenvalues *not* depend on m ?
   (d) Consider a bound state between a proton (charge $+e$) and an anti-proton (charge $-e$), $p\,\bar{p}$; proton mass: $m_p = 1836\,m_e$. In terms of a_0 and E_1, respectively, what is the typical size of $p\,\bar{p}$ in the ground state [see Prob. 5(c)], and what is its dissociation energy ?

2. *Particles in a homogeneous magnetic field.* (**6 points**)
   (a) What are the energy eigenvalues of a free electron in a constant and homogeneous magnetic field B ?
   (b) Consider a hydrogen atom in an energy eigenstate $\varphi_{n\ell m}(\vec{r}) = R_{n\ell}(r)\,Y_\ell^m(\theta, \phi)$. What happens to its energy levels as given in Prob. 1 when a weak (constant) magnetic field $\vec{B}$ is applied ?
   Cyclotron and Larmor frequency: $\Omega = 2\omega_L = eB/m_e c$.

This test sheet has **two** *pages.* **Please turn over !**

3. *Angular momentum commutators.* **(8 points)**
   (a) Determine the commutator $[L_i, p_j]$.
   (b) Write down the uncertainty relation for the product $\Delta L_z \cdot \Delta p_x$ in an arbitrary quantum state $|\psi\rangle$.
   (c) Find $[\vec{L}, \vec{p}^2]$. What does your result imply for the eigenstates of the kinetic energy $T = \vec{p}^2/2m$?

4. *Rigid rotator superposition state.* **(14 points)**
   The normalized angular momentum eigenstates $|\ell\, m\rangle$ are defined via the eigenvalue equations $\vec{L}^2\,|\ell\, m\rangle = \hbar^2\,\ell(\ell+1)\,|\ell\, m\rangle$ and $L_z\,|\ell\, m\rangle = \hbar\, m\,|\ell\, m\rangle$. Moreover, $L_\pm\,|\ell\, m\rangle = \hbar\sqrt{\ell(\ell+1) - m^2 \mp m}\,|\ell\;\; m \pm 1\rangle$ for $L_\pm = L_x \pm i\,L_y$. In position representation, these eigenstates are the spherical harmonics $Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$, $Y_1^0(\theta,\phi) = \sqrt{3/4\pi}\,\cos\theta$, $Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{3/8\pi}\,\sin\theta\, e^{\pm i\phi}$, etc.
   (a) A rigid rotator, $H = \vec{L}^2/2\,I$, is initially prepared in the state $\psi(\theta,\phi,t=0) = C\,(\cos\theta + \sin\theta\cos\phi)$. Determine the constant C.
   (b) In measurements of the energy and L_z in this state, which values will be found, and with what probabilities will they occur ?
   (c) Compute $\langle L_z\rangle_{\psi(0)}$.
   (d) Determine $\psi(\theta,\phi,t)$ for $t > 0$.
   (e) At time $t_0 > 0$, a measurement of L_z results in the value $\hbar$. What is the state $\bar{\psi}(\theta,\phi,t)$ of the system for $t > t_0$?
   (f) Find the expectation value $\langle L_x\rangle_{\bar{\psi}}$ for $t > t_0$.

5. *Hydrogen atom: expectation values.* **(13 points)**
   Consider the (ideal) hydrogen atom ground state wave function $\varphi_{100}(\vec{r}) = A\, e^{-r/a_0}$, with $r = |\vec{r}|$.
   (a) Determine the constant A.
   (b) What is the most likely value of r to be found in a measurement ?
   (c) Compute the expectation values $\langle r\rangle_{100}$ and $\langle z\rangle_{100}$.
   (d) Find $(\Delta r)^2_{100}$ and $(\Delta z)^2_{100}$.

   Useful integrals : $$\int_0^\infty x^k\, e^{-ax}\, dx = \frac{k!}{a^{k+1}}\ .$$

Good luck and keep calm !!!