NAME:

PHYSICS 4456 — INTRODUCTION TO QUANTUM MECHANICS II
Spring 2006 — Second midterm exam — March 28, 2006

Please note: The Virginia Tech *honor code* applies to this midterm. This is a *closed book* exam. Please do *not* communicate with your fellow students ! You may ask for assistance if you find the wording of any of the problem unclear. Please turn in your *entire* work at the end of this test.

Good luck and keep calm !!!

1. *Spin 1/2 dynamics.* **(12 points)**
   The spin-dependent part of the Hamiltonian for a proton (spin $s = 1/2$) in a magnetic field $\vec{B} = B\,\vec{e}_x$ is given by $H = \hbar\omega_L\,\sigma_x$.
   (a) Determine its eigenstates and eigenvalues.
   (b) If the system is initially (at $t = 0$) prepared in the spinor state $|\psi(0)\rangle = \binom{1}{0}$, find $|\psi(t)\rangle = \binom{\psi_+(t)}{\psi_-(t)}$ at $t > 0$.
   (c) For a measurement of S_z at time t, what are the possible results, and with which probabilities will they be found ?
   (d) Compute the expectation values $\langle\sigma_x(t)\rangle_\psi$ and $\langle\sigma_z(t)\rangle_\psi$ at time t.
   Pauli's spin matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

2. *Harmonic oscillator in an electric field.* **(12 points)**
   A particle (charge q, mass m) in a harmonic oscillator potential, with Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2 x^2}{2} = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)$, is subjected to a homogeneous electric field ε, described by the perturbation Hamiltonian $H' = -q\,\varepsilon\,x = -q\,\varepsilon\sqrt{\frac{\hbar}{2m\omega_0}}\left(a + a^\dagger\right)$.
   (a) Confirm the matrix elements $\langle n|x|n'\rangle \propto \sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}$.
   (b) Show that the *first-order* perturbation theory corrections to the energy eigenvalues E_n all vanish.
   (c) Compute the *second-order* energy level shifts. State the condition for the perturbation expansion to yield reliable estimates.
   (d) Find the *exact* energy eigenvalues of $H = H_0 + H'$, and compare with the results from (c).

This test sheet has **two** *pages.* **Please turn over !**

3. *Total angular momentum and spin orbit coupling.* **(14 points)**
Consider the addition of an electron's orbital angular momentum $\vec{L}$ and spin $\vec{S}$, with corresponding quantum numbers ℓ, m_ℓ and $s = 1/2, m_s = \pm 1/2$, resulting in the total angular momentum $\vec{J} = \vec{L} + \vec{S}$, whose quantum numbers we wish to label j and m_j.
   (a) Name the complete set of operators for which the basis states of the *uncoupled representation* are eigenstates.
   How many basis states are in the uncoupled representation ?
   (b) In contrast, the basis states of the *coupled representation* are simultaneous eigenstates for which set of operators ?
   How is m_j related to m_l and m_s ?
   (c) What are the possible values for the quantum number j in terms of ℓ, and, for fixed j, which eigenvalues m_j are allowed ?
   Thus, how many coupled basis states are there ?
   (d) The Hamiltonian for *spin-orbit coupling* is of the form $H_{\rm so} = \epsilon \, \vec{L} \cdot \vec{S}$. Find its eigenvalues, and express them in terms of ℓ.
   (e) What is the order of magnitude for the ensuing fine structure level splitting in hydrogen-like atoms / ions ?

4. *Perturbed two-dimensional infinite box potential.* **(12 points)**
A particle (mass m) confined to a box of width a is governed by the potential $V_a(x) = 0$ for $0 < x < a$, while $V_a(x) = \infty$ for $x \leq 0$ or $x \geq a$.
   (a) Write down the matrix representation for the Hamiltonian of this system in its energy eigenstate basis.
   (b) Consider a *two-dimensional* (square) infinite box, with potential energy $V(x, y) = V_a(x) + V_a(y)$. Provide the energy eigenstates and eigenvalues for the *three lowest* levels.
   (c) For the perturbation Hamiltonian $H' = V_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}$, compute the *first-order energy shifts* for those three lowest levels.

   Recall: $\sin 2x = 2 \sin x \cos x$, and $\cos 2x = \cos^2 x - \sin^2 x$;
   useful integrals ($n \geq 1$):
   $\int (\cos \alpha x)^n \, dx = \frac{1}{n \, \alpha} (\cos \alpha x)^{n-1} \sin \alpha x + \frac{n-1}{n} \int (\cos \alpha x)^{n-2} \, dx$,
   $\int (\sin \alpha x)^n \, dx = -\frac{1}{n \, \alpha} (\sin \alpha x)^{n-1} \cos \alpha x + \frac{n-1}{n} \int (\sin \alpha x)^{n-2} \, dx$.

Good luck and keep calm !!!