NAME:

PHYSICS 4456 — INTRODUCTION TO QUANTUM MECHANICS II
Spring 2007 — First midterm exam — February 15, 2007

Please note: The Virginia Tech *honor code* applies to this midterm. This is a *closed book* exam. Please do *not* communicate with your fellow students ! You may ask me for assistance if you find the wording of any of the problem unclear. Please turn in your *entire* work at the end of this test.

Good luck and keep calm !!!

1. *Hydrogen atom: conceptual questions.* (**8 points**)
   The energy eigenvalues of the (ideal) hydrogen atom bound states are given by $E_n = -R/n^2$, with $n = 1, 2, \ldots$. The Rydberg energy is $R = e^4 \, m_e / 2 \, \hbar^2 = 13.6$ eV.
   (a) For given principal quantum number n, give the allowed values for the angular momentum quantum numbers ℓ and m.
   (b) What is the degeneracy of the nth level E_n (neglecting spin), and why do the energy eigenvalues *not* depend on m ?
   (c) Find the ionization energy of the Li^{++} ion and the dissociation energy of positronium (an e^+–e^- bound state) in the ground state.

2. *Particles in a homogeneous magnetic field.* (**6 points**)
   (a) What are the energy eigenvalues of a free electron in a constant and homogeneous magnetic field B ?
   (b) Consider a hydrogen atom in an energy eigenstate $\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r) \, Y_\ell^m(\theta, \phi)$. What happens to its energy levels as given in Prob. 1 when a weak (constant) magnetic field B is applied ?
   Cyclotron / Larmor frequency: $\omega_c = 2\omega_L = eB/m_e c$.

This test sheet has **two** *pages.* **Please turn over !**

3. *Hydrogen atom: expectation values.* **(14 points)**
Consider the (ideal) hydrogen atom ground state wave function $\psi_{100}(\vec{r}) = A\, e^{-r/a}$, with $r = |\vec{r}|$.
(a) Confirm that $A = 1/\sqrt{\pi\, a^3}$.
(b) What is the most likely value of r to be found in a measurement ?
(c) Compute the expectation values $\langle r^2 \rangle_{100}$, $\langle z \rangle_{100}$, and $\langle z^2 \rangle_{100}$.
(d) Initially, let the atom be prepared in the superposition state $\psi(\vec{r}, t = 0) = \frac{1}{\sqrt{10}} [\psi_{100}(\vec{r}) + 3\, \psi_{310}(\vec{r})]$. Find $\psi(\vec{r}, t)$ at time $t > 0$. What are the probabilities of measuring $L_z = 0$ and $|\vec{L}| = \sqrt{2}\, \hbar$ in this state ?

Useful integrals : $$\int_0^\infty x^k\, e^{-ax}\, dx = \frac{k!}{a^{k+1}} \; .$$

4. *Heisenberg picture.* **(6 points)**
(a) Write down the equation of motion for an operator $O_H(t)$ in the Heisenberg picture.
(b) The Hamiltonian for the one-dimensional harmonic oscillator reads $H = \hbar\, \omega\, (a_+\, a_- + 1/2)$, where $a_\pm = a_\mp^\dagger$ and $[a_-, a_+] = 1$.
Obtain the time dependence of the Heisenberg operator $a_{-H}(t)$.

5. *Spin 1/2 matrix representation.* **(8 points)**
(a) For a spin $s = 1/2$ particle, provide the matrix representations of $\vec{S}^2$, S_z, and S_x in the eigenstate basis of S_z.
(Recall $S_\pm = S_x \pm i S_y$, and $S_+ |\uparrow\rangle = 0$, $S_+ |\downarrow\rangle = \hbar\, |\uparrow\rangle$).
(b) Determine the eigenvalues and eigenspinors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

6. *Homogeneous electric field: momentum representation.* **(8 points)**
A quantum particle with mass m and charge q is placed in a constant electric field $\vec{E} = E_0\, \vec{e}_x$.
(a) Determine the Hamiltonian in the momentum representation.
(b) Solve for the properly normalized energy eigenstates $\phi_E(p)$ in the momentum representation.

Good luck and keep calm !!!