PHYSICS 5455 — QUANTUM MECHANICS I — Fall 2009

Homework assignment 10, due November 3, 2009

The *Graduate Honor Code* applies to this assignment (see homework 1).

1. *Harmonic oscillator coherent states.* **(25 points)**
   Coherent states are the eigenstates of the harmonic oscillator annihilation or lowering operator: $a|\alpha\rangle = \alpha|\alpha\rangle$, with complex eigenvalues α.
   (a) Evaluate the inner product $\langle\alpha|\alpha'\rangle$ between two coherent states, and prove the (over-)completeness relation $\int d^2\alpha \, |\alpha\rangle \, \langle\alpha| = \pi \, 1$.
   Hint: introduce polar coordinates $\alpha = \rho \, e^{i\phi}$ in the complex plane.
   (b) Demonstrate that the temporal evolution of a coherent state can essentially be absorbed into a time-dependent eigenvalue $\alpha(t)$.
   (c) Find the expectation values $\langle x(t)\rangle_\alpha$, $\langle p(t)\rangle_\alpha$, $\langle x^2(t)\rangle_\alpha$, $\langle p^2(t)\rangle_\alpha$, and $\langle H(t)\rangle_\alpha$ in a coherent state, and show that these are consistent with the classical oscillations. Confirm that $(\Delta x)_\alpha \, (\Delta p)_\alpha = \hbar/2$.
   (d) Confirm the representation for time-dependent coherent states $|\alpha(t)\rangle = e^{-i\omega t/2} \exp\left[\alpha(t) \, a^\dagger - \alpha^*(t) \, a\right] |0\rangle$, and derive the explicit expression for the associated wave function $\psi_\alpha(x,t) = \langle x|\alpha(t)\rangle$ provided in the lecture notes.

2. *(*) Anticommuting ladder operators.* **(10 points)**
   Consider ladder operators c and $c^\dagger$ that obey the *anti*commutation relation $\left\{c, c^\dagger\right\} = 1$. Study the eigenvalue spectrum of the corresponding occupation number operator $\hat{n} = c^\dagger \, c$: $\hat{n} \, |n\rangle = n \, |n\rangle$.
   (a) First, find $c \, |n\rangle$ and $c^\dagger \, |n\rangle$ in terms of the eigenstates $|n\rangle$.
   (b) Assuming there exists a vacuum state $|0\rangle$ with $c \, |0\rangle = 0$, which are the only allowed eigenvalues and non-vanishing eigenstates of $\hat{n}$?
   (c) Show that consequently $\left\{c, c\right\} = 0 = \left\{c^\dagger, c^\dagger\right\}$ and $\hat{n}^2 = \hat{n}$.