

Homework assignment 13, due December 1, 2009

The *Graduate Honor Code* applies to this assignment (see homework 1).

1. *Periodic delta function Kronig–Penney potential.* (15 points)

Consider the one-dimensional periodic potential (with  $\lambda > 0$ )

$$V(x) = \lambda \sum_{j=0, \pm 1, \pm 2, \dots} \delta(x - j a) .$$

- Derive the dispersion relation  $\cos(k a) = \cos(\bar{k} a) + \frac{m\lambda}{\hbar^2 \bar{k}} \sin(\bar{k} a)$ .
- Plot the right-hand side of this relation for  $m \lambda a / \hbar^2 = 3\pi/2$  and indicate the resulting allowed energy bands. Show that the upper band edges are located at  $\bar{k} a = n \pi$  with integer  $n = 1, 2, \dots$ .
- Establish that the  $\bar{k}$ -values at the upper and lower band edges respectively correspond to the energy eigenvalues of an infinite / finite single potential well of width  $a$ .

2. (\*) *SUSY partner for the infinite potential well.* (10 points)

Apply supersymmetric quantum mechanics to obtain the energy eigenfunctions and eigenvalues for the potential

$$V(x) = \begin{cases} \frac{\pi^2}{a^2} \left[ \left( \sin \frac{\pi x}{a} \right)^{-2} - \frac{1}{2} \right] & 0 \leq x \leq a \\ \infty & x < 0, x > a \end{cases} .$$

- From the ground-state wave function  $\varphi_0^0(x)$  of the infinite potential well (width  $a$ ), derive the SUSY partner potentials  $V^{0/1}(x)$ .
- Construct the energy eigenfunctions and eigenvalues of  $H^1$  from those of its SUSY partner  $H^0$ .

3. *Angular momentum eigenfunctions and uncertainty.* (15 points)

- Use  $L_+ |\ell \ell\rangle = 0$  in the position representation to compute the normalized eigenfunction  $Y_\ell^\ell(\theta, \phi)$ .
- Therefrom obtain  $Y_\ell^{\ell-1}(\theta, \phi)$  and  $Y_\ell^{\ell-2}(\theta, \phi)$ .
- Calculate  $(\Delta L_x)_{\ell m}$  and  $(\Delta L_y)_{\ell m}$  in the eigenstates  $|\ell m\rangle$  of  $\vec{L}^2$  and  $L_z$ . Show that these uncertainties are minimal for  $|m| = \ell$ .

4. *Oscillator ladder operators and angular momentum.* (10 points)

Write the components of the position and momentum operators in terms of harmonic oscillator ladder operators,  $x_i = \sqrt{\frac{\hbar}{2m\omega}} (a_i + a_i^\dagger)$  and  $p_i = -i\sqrt{\frac{\hbar m\omega}{2}} (a_i - a_i^\dagger)$ , where  $i = x, y$ , and thus express the angular momentum operator  $L_z$  in terms of  $a_i$ . By means of a linear, canonical (Bogoliubov) transformation to new ladder operators  $b_1, b_2$  that satisfy  $[b_i, b_j] = 0$  and  $[b_i, b_j^\dagger] = \delta_{ij}$ , render  $L_z$  in the form  $L_z = \hbar (b_2^\dagger b_2 - b_1^\dagger b_1)$ . Thus establish that the eigenvalues of  $L_z$  are integer:  $m = 0, \pm 1, \pm 2, \dots$