PHYSICS 5455 — QUANTUM MECHANICS I — Fall 2009

Homework assignment 3, due September 15, 2009

The *Graduate Honor Code* applies to this assignment (see homework 1).

1. *Bohr–Sommerfeld quantization.* **(10 points)**

   Apply the Bohr–Sommerfeld quantization rule (in one dimension)

   $$\oint p(x)\,dx = n\,h\ , \qquad n = 1, 2, \ldots$$

   to determine the energy levels for a particle with mass m in

   (a) an infinite potential well of width a: $V(x) = 0$ for $0 < x < a$, whereas $V(x) = \infty$ for $x \leq 0$ and $x \geq a$.

   (b) a harmonic oscillator potential $V(x) = \frac{1}{2}\,m\,\omega^2\,x^2$ (frequency ω).

2. *Dirac's delta distribution.* **(10 + 5 points)**

   Confirm the following properties of the Dirac delta distribution:

   (a) $\delta(a\,x) = \delta(x)/|a|$;

   (b) $f(x)\,\delta'(x) = -f'(0)\,\delta(x)$;

   (c) $\delta(-x) = \delta(x)$ and $\delta'(-x) = -\delta'(x)$;

   (d) if x_i denote the (simple) zeros of the function F, i.e., $F(x_i) = 0$:

   $$\delta\big(F(x)\big) = \sum_i \frac{\delta(x - x_i)}{|F'(x_i)|}\ .$$

   Thus evaluate $\delta(x^2 - a^2)$.

   (e) (*) *Extra-credit assignment:*

   Establish the Fourier representation of the Heaviside step function

   $$\Theta(x) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{k - i\epsilon}\, dk\ ,$$

   and consequently

   $$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk\ .$$

3. *One-dimensional wave packet.* **(10 points)**
Consider a one-dimensional wave packet with $\phi(p) = A\,\Theta(\hbar/d - |p - p_0|)$.
(a) Determine the constant A and find the spatial wave function $\psi(x)$ (ignore the temporal evolution, i.e., set $t = 0$).
(b) Compute the expectation values $\langle x \rangle$, $\langle p \rangle$, and $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

4. *Independent subsystems and probability interpretation.* **(10 points)**
Consider the time-dependent Schrödinger equation

$$i\hbar\,\frac{\partial\,\psi(t)}{\partial t} = H\,\psi(t)$$

with a Hamiltonian that describes two non-interacting, independent subsystems: $H = H_1 + H_2$, with $[H_1, H_2] = 0$.
(a) Demonstrate that its solution can be written as a product wave function $\psi(t) = \psi_1(t)\,\psi_2(t)$, where $\psi_{1/2}(t)$ solve the time-dependent Schrödinger equation for the subsystem Hamiltonians $H_{1/2}$.
(b) Why is this result consistent with Born's probability amplitude interpretation of the wave function ?
(c) Now study the modified wave equation $\kappa\,\partial^2\psi/\partial t^2 = H\,\psi$. In the case of non-interacting subsystems, would a similar factorization as in (a) work here as well ?