PHYSICS 5455 — QUANTUM MECHANICS I — Fall 2009

Homework assignment 4, due September 22, 2009

The *Graduate Honor Code* applies to this assignment (see homework 1).

1. *Operator identities.* (**20 points**)
   For operators A, B, C, show that
   (a) $(A\,B)^\dagger = B^\dagger\,A^\dagger$;
   (b) $[A,B]^\dagger = [B^\dagger, A^\dagger]$;
   (c) $[A\,B, C] = A\,[B,C] + [A,C]\,B$;
   (d) $[A, B^k] = k\,B^{k-1}\,[A,B]$, provided $\big[[A,B],B\big] = 0$;
   (e) $e^A\,e^B = e^B\,e^A\,e^{[A,B]}$ and $e^{A+B} = e^A\,e^B\,e^{-[A,B]/2}$, if $\big[[A,B],A\big] = 0 = \big[[A,B],B\big]$.
   Hint: consider the λ-derivative of the functions $e^{\lambda A}e^{\lambda B}$ and $e^{\lambda(A+B)}$.
   (f) Prove the *Baker–Hausdorff identity*
   $$e^A\,B\,e^{-A} = B+[A,B]+\frac{1}{2}\Big[A,[A,B]\Big]+\ldots+\frac{1}{k!}\Big[A,\Big[A,\ldots[A,B]\ldots\Big]\Big]+\ldots$$

2. *Continuity equation for the Gaussian wave packet.* (**10 points**)
   Confirm that the one-dimensional Gaussian wave packet discussed in class satisfies the continuity equation for probability conservation.

3. *Schrödinger equation Green function / propagator.* (**10 + 5 points**)
   (a) Show that a general solution of the time-dependent Schrödinger equation can be written in terms of the initial wave function at $t = 0$ in the form $\psi(\vec{r},t) = \int K(\vec{r},\vec{r}\,';t)\,\psi(\vec{r}\,',0)\,d^dr'$, where the *propagator* (Green function) $K(\vec{r},\vec{r}\,';t)$ satisfies the time-dependent Schrödinger equation, $i\hbar\,\partial K(\vec{r},\vec{r}\,';t)/\partial t = H(\vec{r})\,K(\vec{r},\vec{r}\,';t)$, with initial condition $K(\vec{r},\vec{r}\,';0) = \delta(\vec{r}-\vec{r}\,')$.
   (b) Demonstrate that $K(\vec{r},\vec{r}\,';t) = \int K(\vec{r},\vec{s};t-t')\,K(\vec{s},\vec{r}\,';t')\,d^ds$.
   (c) Prove the representation of the propagator in terms of energy eigenfunctions, $H\,\varphi_n(\vec{r}) = E_n\,\varphi_n(\vec{r})$ (provided $\partial H/\partial t = 0$):
   $$K(\vec{r},\vec{r}\,';t) = \sum_n \varphi_n^*(\vec{r}\,')\,\varphi_n(\vec{r})\,e^{-iE_nt/\hbar}\ .$$
   (d) (*) Evaluate the propagator explicitly for a free particle (in d dimensions; in intermediate steps, use imaginary time $\tau = it$).