

Homework assignment 9, due October 27, 2009

The *Graduate Honor Code* applies to this assignment (see homework 1).

1. *Hermite polynomials*. (15 points)

The Hermite polynomials H_n are defined via the generating function

$$e^{-t^2+2ty} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(y) .$$

(a) Thus, establish the explicit derivative representations

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2} = e^{y^2/2} \left(y - \frac{d}{dy} \right)^n e^{-y^2/2} .$$

For the second expression, first prove the operator identity $e^{-y^2/2} (y - d/dy)^n e^{y^2/2} = (-1)^n d^n/dy^n$.

(b) Confirm the two recursion relations $H'_n(y) = 2n H_{n-1}(y)$ and $H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y)$, and write down the first six Hermite polynomials (H_0, H_1, \dots, H_5). Show that the H_n satisfy Hermite's differential equation $H''_n(y) - 2y H'_n(y) + 2n H_n(y) = 0$.

2. (*) *Harmonic oscillator eigenfunctions: orthogonality, completeness*. (15 points)

The harmonic oscillator energy eigenfunctions ($n = 0, 1, 2, \dots$) read

$$\varphi_n(x) = \sqrt{\frac{1}{2^n n! \sqrt{\pi} x_0}} H_n(x/x_0) e^{-x^2/2x_0^2} , \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} .$$

(a) Derive the orthogonality relation for Hermite polynomials,

$$\int e^{-y^2} H_n(y) H_{n'}(y) dy = 2^n n! \sqrt{\pi} \delta_{nn'} ,$$

which establishes the orthonormality of the eigenfunctions φ_n .

(b) Through explicit calculation, confirm the completeness relation

$$\sum_{n=0}^{\infty} \varphi_n(x) \varphi_n(x') = \delta(x - x') .$$

Hint: utilize the Fourier representation of a Gauss function, $2\sqrt{\pi} e^{-x^2} = \int e^{ikx - k^2/4} dk$.

3. *Harmonic oscillator expectation values.* (15 points)

The Hamiltonian for the one-dimensional harmonic oscillator reads

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right),$$

where $x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$ and $p = -i\sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger)$.

- (a) Evaluate the matrix elements $x_{nn'} = \langle n|x|n'\rangle$ and $p_{nn'} = \langle n|p|n'\rangle$. Mapping the energy eigenstates $|n\rangle$ to Cartesian unit vectors, write down an explicit matrix representation for the operators x and p in the energy basis, and show that $\langle x \rangle_n = 0 = \langle p \rangle_n$.
- (b) Find the corresponding matrices for the operators x^2 , p^2 , and H .
- (c) Compute $\langle x^2 \rangle_n$, $\langle p^2 \rangle_n$, and the uncertainty product $(\Delta x)_n (\Delta p)_n$. Establish the virial theorem $\langle T \rangle_n = \langle V \rangle_n = E_n/2$.

4. *Harmonic oscillator variants.* (10 points)

Find the energy eigenvalues and eigenfunctions for the following two variants of the harmonic oscillator Hamiltonian (*no explicit solution of any differential equation is required here!*):

- (a) Charged particle in a harmonic oscillator potential and a constant electric field E_0 : $V(x) = m\omega^2 x^2/2 - qE_0 x$.
- (b) Harmonic oscillator in half-space:
 $V(x) = m\omega^2 x^2/2$ for $x > 0$, $V(x) = \infty$ for $x \leq 0$.

Second midterm exam: **Thursday, Oct. 29, 5.00 pm** (in class)

- Closed-book: no books, notes, calculators, electronic devices permitted.
- Please answer ALL questions, provide brief explanations.
- Material covered: Chaps. 2.1-4 and 3.1-2 in the syllabus, pp. 27–56 in the lecture notes, homework assignments 5 through 9 (not the extra-credit problems).
- The graduate honor code pertains to the midterm exam.