Field Theory Approaches to Nonequilibrium Dynamics

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1 Critical dynamics

1.1 Introduction: critical slowing down

Near a *critical point* $\tau = (T - T_c)/T_c \ll 1$: large fluctuations, induce diverging *correlation length* $\xi(\tau) \sim |\tau|^{-\nu}$, *scaling laws*:

$$C(\tau, q) = |q|^{-2+\eta}\, \hat{C}_\pm(q\,\xi) \ , \quad C(\tau, x) = |x|^{-(d-2+\eta)}\, \widetilde{C}_\pm(x/\xi) \ ;$$

$\Longrightarrow$ *two* independent static *critical* exponents ν, η.
Other power laws, e.g., *order parameter* S, *susceptibility* χ_τ:

$$|x| \to \infty: \ C(\tau, x) \propto \xi^{-(d-2+\eta)} \sim |\tau|^{\nu(d-2+\eta)} \propto \langle S \rangle^2 \sim |\tau|^{2\beta} \ ,$$
$$\chi_\tau(\tau) \propto \lim_{q \to 0} C(\tau, q) \propto \xi^{2-\eta} \sim |\tau|^{-\nu(2-\eta)} = |\tau|^{-\gamma}$$
$$\Longrightarrow \quad \beta = \frac{\nu}{2}\,(d - 2 + \eta) \ , \quad \gamma = \nu\,(2 - \eta) \ .$$

Mathematical description for $O(n)$-symmetric order parameter: *canonical* distribution with *Landau–Ginzburg–Wilson* functional

$$\begin{aligned} &\mathcal{P}_{\rm eq}[S] \propto \exp\left(-\mathcal{H}[S]/k_{\rm B}T\right) \ , \\ &\mathcal{H}[S] = \int d^dx \sum_\alpha \left[\frac{r}{2}[S^\alpha(x)]^2 + \frac{1}{2}[\nabla S^\alpha(x)]^2 \right. \\ &\qquad \left. + \frac{u}{4!}\sum_\beta [S^\alpha(x)]^2 [S^\beta(x)]^2 - h^\alpha(x) S^\alpha(x)\right] , \end{aligned} \tag{1}$$

control parameters: transition driven by non-linearity u, $r \propto T - T_c^0$ (mean-field), h external field conjugate to S.

Dynamics: expect *slow relaxation* for large correlated regions, characteristic time scale defines *dynamic critical exponent* z:

$$t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu} \ , \quad \omega_c(\tau) \sim |\tau|^{z\nu} \ .$$

Dynamic scaling hypothesis: $t_c(\tau, q)^{-1} = \omega_c(\tau, q) = |q|^z\, \hat{\omega}_\pm(q\,\xi)$

$$\Longrightarrow \ \langle S(\tau, t)\rangle = |\tau|^\beta\, \hat{S}(t/t_c) \ , \quad \chi(\tau, q, \omega) = |q|^{-2+\eta}\, \hat{\chi}_\pm(q\,\xi, \omega\,\xi^z) \ .$$

Apply *equilibrium* fluctuation-dissipation theorem:

$$C(\tau, q, \omega) = \frac{2k_{\rm B}T}{\omega} \, {\rm Im} \, \chi(\tau, q, \omega) = |q|^{-z-2+\eta} \, \hat{C}_\pm \left(q \, \xi, \omega \, \xi^z \right) \ ,$$
$$C(\tau, x, t) = |x|^{-(d-2+\eta)} \, \widetilde{C}_\pm \left(x/\xi, t/\xi^z \right) \ .$$

Coarse-grained description: fast modes $\to$ *random* 'noise', $\Longrightarrow$ *mesoscopic Langevin* equation for *slow* variables $S^\alpha(x, t)$.
Example: purely *relaxational* critical dynamics, *models A / B* :

$$\frac{\partial S^\alpha(x, t)}{\partial t} = -D \, \frac{\delta \mathcal{H}[S]}{\delta S^\alpha(x, t)} + \zeta^\alpha(x, t) \ , \quad \langle \zeta^\alpha(x, t) \rangle = 0 \ ,$$
$$\left\langle \zeta^\alpha(x, t) \, \zeta^\beta(x', t') \right\rangle = 2D \, k_{\rm B}T \, \delta(x - x') \delta(t - t') \delta^{\alpha\beta} \ ; \qquad (2)$$

Einstein relation guarantees that $\mathcal{P}[S, t] \to \mathcal{P}_{\rm eq}[S]$ as $t \to \infty$.
Conserved order parameter: relaxes *diffusively*, $D \to -D \, \nabla^2$;
model A / B: $D(i\nabla)^a$, $a = 0, 2$: conserved / non-conserved.
Generally: mode-couplings to other *conserved*, slow fields; splitting into several *dynamic universality classes*.
Gaussian (mean-field) approximation: $u = 0$

$$\left[-i\omega + Dq^a(r + q^2) \right] S^\alpha(q, \omega) = Dq^a \, h(q, \omega) + \zeta^\alpha(q, \omega) \ ,$$
$$\langle \zeta^\alpha(q, \omega) \zeta^\beta(q', \omega') \rangle = 2k_{\rm B}TDq^a (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta},$$

$\Longrightarrow$ dynamic response and correlation functions:

$$\chi_0^{\alpha\beta}(q, \omega) = \partial \langle S^\alpha(q, \omega) \rangle / \partial h^\beta(q, \omega)|_{h=0} = Dq^a \, G_0(q, \omega) \, \delta^{\alpha\beta} \ ,$$
$$G_0(q, \omega) = \left[-i\omega + Dq^a(r + q^2) \right]^{-1} \ ,$$
$$\left\langle S^\alpha(q, \omega) S^\beta(q', \omega') \right\rangle_0 = C_0(q, \omega) (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta},$$
$$C_0(q, \omega) = \frac{2k_{\rm B}TDq^a}{\omega^2 + [Dq^a(r + q^2)]^2} = 2k_{\rm B}TDq^a \, |G_0(q, \omega)|^2 \ ;$$
$$G_0(q, t) = \Theta(t) \, e^{-Dq^a \, (r+q^2) \, t} \ , \ C_0(q, t) = \frac{k_{\rm B}T}{r + q^2} \, e^{-Dq^a \, (r+q^2) \, |t|} \ .$$

$\Longrightarrow$ *Gaussian critical exponents*: $\eta_0 = 0$, $\nu_0 = \frac{1}{2}$, $z_0 = 2 + a$.

1.2 Field theory representation of Langevin equations

Coupled Langevin equations for mesoscopic stochastic variables:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = F^\alpha[S](x,t) + \zeta^\alpha(x,t) \ , \quad \langle \zeta^\alpha(x,t) \rangle = 0 \ ,$$
$$\left\langle \zeta^\alpha(x,t) \zeta^\beta(x',t') \right\rangle = 2L^\alpha \, \delta(x-x') \, \delta(t-t') \, \delta^{\alpha\beta} \ ;$$

$F^\alpha[S]$: 'systematic forces', ζ^α: 'stochastic forces (*noise*)', correlator L^α: can be differential operator, functional of S^α. Assume *Gaussian* stochastic process, probability distribution:

$$\mathcal{W}[\zeta] \propto \exp \left\{ -\frac{1}{4} \int d^d x \int_0^{t_f} dt \sum_\alpha \zeta^\alpha(x,t) \left[(L^\alpha)^{-1} \zeta^\alpha(x,t) \right] \right\} \ ,$$

switch variables $\zeta^\alpha \to S^\alpha$: $\mathcal{W}[\zeta] \, \mathcal{D}[\zeta] = \mathcal{P}[S] \mathcal{D}[S] \propto e^{-\mathcal{G}[S]} \mathcal{D}[S]$, with *Onsager-Machlup* functional providing field-theory *action*:

$$\mathcal{G}[S] = \frac{1}{4} \int d^d x \int dt \sum_\alpha \left(\frac{\partial S^\alpha}{\partial t} - F^\alpha[S] \right) \left[(L^\alpha)^{-1} \left(\frac{\partial S^\alpha}{\partial t} - F^\alpha[S] \right) \right] \ ;$$

functional determinant $= 1$ with *forward* (Ito) discrectization, normalization: $\int \mathcal{D}[\zeta] W[\zeta] = 1 \Longrightarrow$ 'partition function' $= 1$; problems: $(L^\alpha)^{-1}$, high non-linearities $F^\alpha[S] \, (L^\alpha)^{-1} \, F^\alpha[S]$.

Goal: average noise 'histories': $\langle A[S] \rangle_\zeta \propto \int \mathcal{D}[\zeta] \, A[S(\zeta)] \, W[\zeta]$:

$$1 = \int \mathcal{D}[S] \prod_\alpha \prod_{(x,t)} \delta\Big(\partial_t \, S^\alpha(x,t) - F^\alpha[S](x,t) - \zeta^\alpha(x,t) \Big)$$
$$= \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp \Big[- \int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha \left(\partial_t \, S^\alpha - F^\alpha[S] - \zeta^\alpha \right) \Big] \ ,$$
$$\langle A[S] \rangle_\zeta \propto \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp \Big[- \int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha \left(\partial_t S^\alpha - F^\alpha[S] \right) \Big]$$
$$\times \, A[S] \int \mathcal{D}[\zeta] \exp \left(- \int d^d x \int dt \sum_\alpha \left[\frac{1}{4} \, \zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha - \widetilde{S}^\alpha \, \zeta^\alpha \right] \right) ;$$

perform Gaussian integral over noise ζ^α:

$$\langle A[S] \rangle_\zeta = \int \mathcal{D}[S] \, A[S] \, \mathcal{P}[S] \ , \quad \mathcal{P}[S] \propto \int \mathcal{D}[i\widetilde{S}] \, e^{-\mathcal{A}[\widetilde{S},S]} \ ,$$

with *Janssen–De Dominicis* 'response' functional:

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int_0^{t_f} dt \sum_\alpha \left[\widetilde{S}^\alpha \left(\partial_t S^\alpha - F^\alpha[S]\right) - \widetilde{S}^\alpha L^\alpha \widetilde{S}^\alpha\right] \ , \quad (3)$$

$\int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \, e^{-\mathcal{A}[\widetilde{S},S]} = 1$; integrate out $\widetilde{S}^\alpha \rightarrow$ Onsager–Machlup.
Relaxational models A / B: $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{\rm int}$, with ($k_{\rm B}T = 1$):

$$\mathcal{A}_0[\widetilde{S}, S] = \int d^d x \int dt \sum_\alpha \Big(\widetilde{S}^\alpha \left[\partial_t + D(i\nabla)^a (r - \nabla^2)\right] S^\alpha - D\widetilde{S}^\alpha (i\nabla)^a \widetilde{S}^\alpha - D\widetilde{S}^\alpha (i\nabla)^a h^\alpha \Big) \ ,$$

$$\mathcal{A}_{\rm int}[\widetilde{S}, S] = D \, \frac{u}{6} \int d^d x \int dt \sum_{\alpha,\beta} \widetilde{S}^\alpha (i\nabla)^a S^\alpha S^\beta S^\beta \ ;$$

$$\chi^{\alpha\beta}(x-x', t-t') = \frac{\delta \langle S^\alpha(x,t) \rangle}{\delta h^\beta(x',t')}\Big|_{h=0} = D \left\langle S^\alpha(x,t) (i\nabla)^a \, \widetilde{S}^\beta(x',t') \right\rangle \ ;$$

$\Longrightarrow$ $\widetilde{S}^\alpha$ 'response' fields; fluctuation-dissipation theorem:

$$\chi^{\alpha\beta}(x - x', t - t') = \Theta(t - t') \, \frac{\partial}{\partial t'} \left\langle S^\alpha(x,t) S^\beta(x',t') \right\rangle \ .$$

Generating functional for correlation functions, *cumulants*:

$$\mathcal{Z}[\tilde{j}, j] = \left\langle \exp \int d^d x \int dt \sum_\alpha \left(\tilde{j}^\alpha \, \widetilde{S}^\alpha + j^\alpha \, S^\alpha \right) \right\rangle \ ,$$

$$\left\langle \prod_{ij} S^{\alpha_i} \, \widetilde{S}^{\alpha_j} \right\rangle_{(c)} = \prod_{ij} \frac{\delta}{\delta j^{\alpha_i}} \, \frac{\delta}{\delta \tilde{j}^{\alpha_j}} \, (\ln) \mathcal{Z}[\tilde{j}, j] \Big|_{\tilde{j}=j=0} \ .$$

Harmonic approximation ($u = 0$): Gaussian integration recovers

$$\left\langle S^\alpha(q,\omega) \widetilde{S}^\beta(q',\omega') \right\rangle_0 = G_0(q,\omega) (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta},$$

$$G_0(q,\omega) = \left[-i\omega + Dq^a (r + q^2)\right]^{-1} \ ,$$

$$\left\langle S^\alpha(q,\omega) S^\beta(q',\omega') \right\rangle_0 = C_0(q,\omega) (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta},$$

$$C_0(q,\omega) = \frac{2Dq^a}{\omega^2 + [Dq^a (r + q^2)]^2} = 2Dq^a \, |G_0(q,\omega)|^2 \ ;$$

causality: $\left\langle \widetilde{S}^\alpha(q,\omega) \widetilde{S}^\beta(q',\omega') \right\rangle_0 = 0$.

1.3 Outline of dynamic perturbation theory

Non-linear terms $\sim u$: treat by means of *perturbation* expansion:

$$\left\langle \prod_{ij} S^{\alpha_i} \, \widetilde{S}^{\alpha_j} \right\rangle = \frac{\left\langle \prod_{ij} S^{\alpha_i} \, \widetilde{S}^{\alpha_j} \, e^{-\mathcal{A}_{\rm int}[\widetilde{S},S]} \right\rangle_0}{\left\langle e^{-\mathcal{A}_{\rm int}[\widetilde{S},S]} \right\rangle_0}$$

$$= \left\langle \prod_{ij} S^{\alpha_i} \, \widetilde{S}^{\alpha_j} \sum_{l=0}^{\infty} \frac{1}{l!} \left(-\mathcal{A}_{\rm int}[\widetilde{S},S] \right)^l \right\rangle_0 \, ,$$

since the denominator $= 1$: *no 'vacuum'* contributions.
Graphical representation: *directed* (causality !) *propagator lines* connect $\widetilde{S}^\beta$ to S^α, join at *vertices* (subject to q, ω conservation):

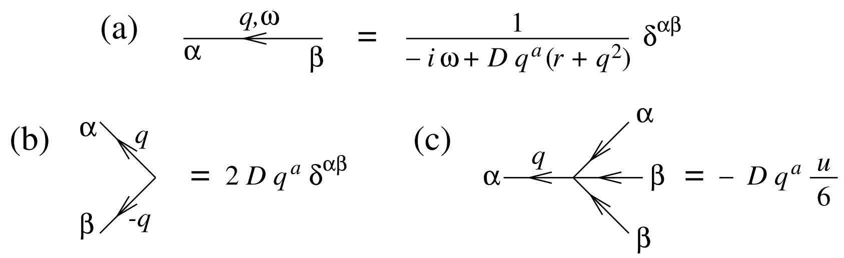

Figure 1: Elements of dynamic perturbation theory for the $O(n)$-symmetric relaxational models: (a) propagator; (b) noise vertex; (c) anharmonic vertex.

Follow standard procedures: *cumulants* $\longleftrightarrow$ *connected* graphs;
Dyson equation for propagator: $G(q,\omega)^{-1} = G_0(q,\omega)^{-1} - \Sigma(q,\omega)$.
$\widetilde{\Phi}^\alpha = \delta \ln \mathcal{Z} / \delta \tilde{j}^\alpha$, $\Phi^\alpha = \delta \ln \mathcal{Z} / \delta j^\alpha \implies$ *generating functional*:

$$\Gamma[\widetilde{\Phi}, \Phi] = - \ln \mathcal{Z}[\tilde{j}, j] + \int d^d x \int dt \sum_\alpha \left(\tilde{j}^\alpha \, \widetilde{\Phi}^\alpha + j^\alpha \, \Phi^\alpha \right) \, ,$$

$$\Gamma^{(\widetilde{N},N)}_{\{\alpha_i\};\{\alpha_j\}} = \prod_i^{\widetilde{N}} \frac{\delta}{\delta \widetilde{\Phi}^{\alpha_i}} \prod_j^{N} \frac{\delta}{\delta \Phi^{\alpha_j}} \, \Gamma[\widetilde{\Phi}, \Phi] \Big|_{\tilde{j}=0=j} \implies$$

$$\Gamma^{(1,1)}(q,\omega) = G(-q,-\omega)^{-1} \, , \quad \Gamma^{(0,2)}(q,\omega) = 0 \quad \text{(causality)} \, ,$$

$$\Gamma^{(2,0)}(q,\omega) = - \frac{C(q,\omega)}{|G(q,\omega)|^2} = - \frac{2Dq^a}{\omega} \, {\rm Im} \, \Gamma^{(1,1)}(q,\omega) \quad \text{(FDT)} \, ,$$

$\implies$ *vertex functions* $\longleftrightarrow$ *one-particle* (1PI) *irreducible graphs.*

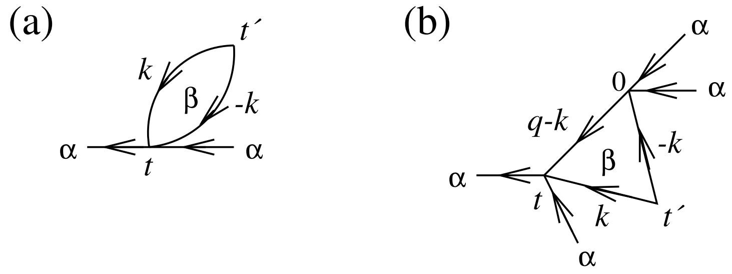

Figure 2: One-loop diagrams for (a) $\Gamma^{(1,1)}$ and (b) $\Gamma^{(1,3)}$ in the time domain.

Feynman rules for the l-th order contribution to $\Gamma^{(\widetilde{N},N)}$:

1. Draw all topologically different, connected *one-particle irreducible graphs* with $\widetilde{N}$ outgoing and N incoming lines connecting l relaxation vertices $\propto u$.
   Do *not* allow closed response loops (Ito calculus: $\Theta(0) = 0$).

2. Attach wavevectors q_i, frequencies ω_i or times t_i, and vector indices α_i to all directed lines, obeying 'momentum (and energy)' conservation at each vertex.

3. Each directed line corresponds to a response propagator $G_0(-q, -\omega)$ / $G_0(q, t_i - t_j)$, the two-point vertex to the noise strength $2Dq^a$, and the four-point relaxation vertex to $-Dq^a u/6$. Closed loops imply integrals over the internal wavevectors and frequencies or times, subject to causality constraints, as well as sums over the internal vector indices. Apply residue theorem to evaluate frequency integrals.

4. Multiply with -1 and the combinatorial factor counting all possible ways of connecting the propagators, l relaxation vertices, and k two-point vertices leading to topologically identical graphs, including a factor $1/l!\,k!$ originating in the expansion of $\exp(-\mathcal{A}_{\text{int}}[\widetilde{S}, S])$.

Perturbation series $\Longrightarrow$ *loop expansion*, $\Delta(q) = Dq^a(r + q^2)$:

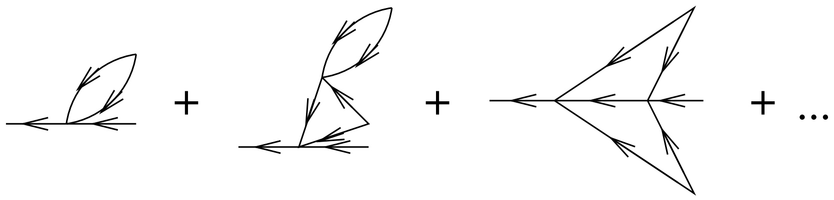

Figure 3: 1PI diagrams for $\Sigma(q,\omega)$ or $\Gamma^{(1,1)}(q,\omega)$ to second order in u.

$$\Gamma^{(1,1)}(q,\omega) = i\omega + Dq^a\Big[r + q^2 + \frac{n+2}{6}\,u\int_k \frac{1}{r+k^2} - \left(\frac{n+2}{6}\,u\right)^2 \int_k \frac{1}{r+k^2}\int_{k'}\frac{1}{(r+k'^2)^2} - \frac{n+2}{18}\,u^2\int_k\frac{1}{r+k^2}\int_{k'}\frac{1}{r+k'^2}\,\frac{1}{r+(q-k-k')^2} \times\left(1 - \frac{i\omega}{i\omega+\Delta(k)+\Delta(k')+\Delta(q-k-k')}\right)\Big] .$$

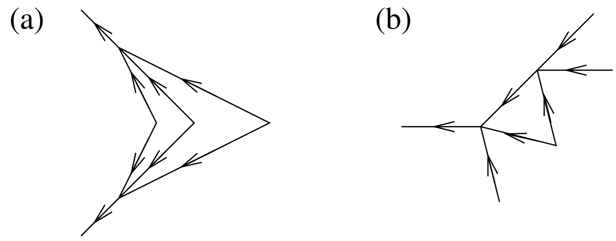

Figure 4: (a) Two-loop diagram for $\Gamma^{(2,0)}(q,\omega)$; (b) one-loop graph for $\Gamma^{(1,3)}$.

$$\Gamma^{(2,0)}(q,\omega) = -2Dq^a\Big[1 + Dq^a\,\frac{n+2}{18}\,u^2\int_k\frac{1}{r+k^2}\int_{k'}\frac{1}{r+k'^2} \times\frac{1}{r+(q-k-k')^2}\,\mathrm{Re}\,\frac{1}{i\omega+\Delta(k)+\Delta(k')+\Delta(q-k-k')}\Big] ,$$

$$\underline{k} = (q,\omega):\ \Gamma^{(1,3)}(-3\underline{k}/2;\{\underline{k}/2\}) = D\left(\frac{3}{2}\,q\right)^a u\Big[1 - \frac{n+8}{6}\,u \times\int_k\frac{1}{r+k^2}\,\frac{1}{r+(q-k)^2}\left(1-\frac{i\omega}{i\omega+\Delta(k)+\Delta(q-k)}\right)\Big] ;$$

$$a = 2\ :\ \Gamma^{(1,1)}(q=0,\omega) = i\omega\ ,\quad \partial_{q^2}\,\Gamma^{(2,0)}(q,\omega)\big|_{q=0} = -2D\ .$$

1.4 Renormalization, scaling laws, critical exponents

$$d < 4 \ : \ u \int \frac{d^d k}{(2\pi)^d} \frac{1}{(r + k^2)^2} = \frac{u \, r^{-2+d/2}}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\infty \frac{x^{d-1}}{(1 + x^2)^2} \, dx \ ,$$

effective coupling $u \, r^{(d-4)/2} \to \infty$ as $r \to 0$: *infrared* divergence,
$\Longrightarrow$ fluctuation corrections singular, modify critical power laws;

$$\int_0^\Lambda \frac{k^{d-1}}{(r + k^2)^2} \, dk \sim \left\{ \begin{array}{cc} \ln(\Lambda^2/r) & d = 4 \\ \Lambda^{d-4} & d > 4 \end{array} \right\} \to \infty \quad \text{as } \Lambda \to \infty \ ,$$

ultraviolet divergences for $d > d_c = 4$: *upper critical dimension.*

dimension interval	perturbation series	model A / B or Φ^4 field theory	critical behavior
$d \leq d_{lc} = 2$	IR-singular UV-convergent	ill-defined u relevant	no long-range order ($n \geq 2$)
$2 < d < 4$	IR-singular UV-convergent	super-renormalizable u relevant	non-classical exponents
$d = d_c = 4$	logarithmic IR-/ UV-divergence	renormalizable u marginal	logarithmic corrections
$d > 4$	IR-regular UV-divergent	non-renormalizable u irrelevant	mean-field exponents

Table 1: Mathematical and physical distinctions of the regimes $d < d_c$, $d = d_c$, and $d > d_c$, for the $O(n)$-symmetric models A / B (Φ^4 field theory).

Power counting in terms of arbitrary momentum scale μ:
$[x] = \mu^{-1}$, $[q] = \mu$, $[t] = \mu^{-2-a}$, $[\omega] = \mu^{2+a} \Longrightarrow [D] = \mu^0$,
$[r] = \mu^2 \to$ *relevant*, $[u] = \mu^{4-d}$ *marginal* at $d_c = 4$.
Integrals in *dimensional regularization*: even for non-integer d, σ:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\sigma}}{(\tau + k^2)^s} = \frac{\Gamma(\sigma + d/2) \, \Gamma(s - \sigma - d/2)}{2^d \, \pi^{d/2} \, \Gamma(d/2) \, \Gamma(s)} \ \tau^{\sigma - s + d/2} \ . \quad (4)$$

Renormalization program:
(i) Keep track of UV singularities as poles in $\epsilon = d_c - d$;
(ii) infer scaling properties under change of momentum scale μ;
(iii) at scale-invariant fixed point: extract IR power laws.

Additive renormalization, fluctuation-induced T_c shift:
inverse static susceptibility $\chi(0,0)^{-1} = \tau = r - r_c$

$$\Longrightarrow r_c = -\frac{n+2}{6}\, u \int_k \frac{1}{r_c + k^2} + O(u^2) = -\frac{n+2}{6} \frac{u\, K_d}{(2\pi)^d} \frac{\Lambda^{d-2}}{d-2} \, ,$$

$$\chi(q,\omega)^{-1} = -\frac{i\omega}{D q^a} + q^2 + \tau \left[1 - \frac{n+2}{6}\, u \int_k \frac{1}{k^2(\tau + k^2)} \right] .$$

Multiplicative renormalization: absorb UV poles at $\epsilon = 0$ into $S^\alpha_R = Z_S^{1/2}\, S^\alpha$, $\widetilde{S}^\alpha_R = Z_{\widetilde{S}}^{1/2} \widetilde{S}^\alpha \Longrightarrow \Gamma_R^{(\widetilde{N},N)} = Z_{\widetilde{S}}^{-\widetilde{N}/2} Z_S^{-N/2}\, \Gamma^{(\widetilde{N},N)}$, and $D_R = Z_D D$, $\tau_R = Z_\tau \tau\, \mu^{-2}$, $u_R = Z_u u\, A_d\, \mu^{d-4}$.
FDT: $Z_D = \left(Z_S / Z_{\widetilde{S}}\right)^{1/2}$, $\chi_R = Z_S\, \chi$; $a = 2$: $Z_{\widetilde{S}} Z_S = 1$, $Z_D = Z_S$.
Normalization point outside IR regime, e.g., $\tau_R = 1$ or $q = \mu$;
one-loop: $Z_S = 1$, $Z_D = 1$, and with $A_d = \Gamma(3 - d/2)/2^{d-1}\, \pi^{d/2}$:

$$Z_\tau = 1 - \frac{n+2}{6} \frac{u_R}{\epsilon} \, , \quad Z_u = 1 - \frac{n+8}{6} \frac{u_R}{\epsilon} \, ;$$

two-loop: from $\partial_{q^2}\, \chi_R(q,0)|_{q=0}$ and $\Gamma_R^{(2,0)}(0,0)$ or $\Gamma_R^{(0,0)}(0,\omega)$:

$$Z_S = 1 + \frac{n+2}{144} \frac{u_R^2}{\epsilon} \, , \quad Z_D = 1 - \frac{n+2}{144} \left(6 \ln \frac{4}{3} - 1 \right) \frac{u_R^2}{\epsilon} \, .$$

Unrenormalized quantities can*not* depend on scale μ:

$$0 = \mu \frac{d}{d\mu} \Gamma^{(\widetilde{N},N)} \Big|_{D,\tau,u} = \mu \frac{d}{d\mu} \left[Z_{\widetilde{S}}^{\widetilde{N}/2} Z_S^{N/2}\, \Gamma_R^{(\widetilde{N},N)}(\mu, D_R, \tau_R, u_R) \right]$$

$\Longrightarrow$ *Callan–Symanzik RG* equation: $\delta\, \Gamma_R^{(\widetilde{N},N)} = 0$, with

$$\delta = \mu \frac{\partial}{\partial \mu} + \frac{\widetilde{N} \gamma_{\widetilde{S}} + N \gamma_S}{2} + \gamma_D D_R \frac{\partial}{\partial D_R} + \gamma_\tau \tau_R \frac{\partial}{\partial \tau_R} + \beta_u \frac{\partial}{\partial u_R} \, ; \quad (5)$$

Wilson's flow functions: $\gamma_{\widetilde{S}} = \mu\, \partial_\mu|_0 \ln Z_{\widetilde{S}}$, $\gamma_S = \mu\, \partial_\mu|_0 \ln Z_S$; $\gamma_D = \mu\, \partial_\mu|_0 \ln(D_R/D) = \frac{1}{2} \left(\gamma_S - \gamma_{\widetilde{S}}\right)$; model B: $\gamma_D = \gamma_S = -\gamma_{\widetilde{S}}$; $\gamma_\tau = \mu\, \partial_\mu|_0 \ln(\tau_R/\tau) = -2 + \mu\, \partial_\mu|_0 \ln Z_\tau$, and
RG beta function $\beta_u = \mu\, \partial_\mu|_0 u_R = u_R\, (d - 4 + \mu\, \partial_\mu|_0 \ln Z_u)$.

Dynamic susceptibility $\chi(q,\omega) = Dq^a\, \Gamma^{(1,1)}(-q,-\omega)^{-1}$:
$\chi_R(\mu, D_R, \tau_R, u_R, q, \omega)^{-1} = \mu^2\, \hat{\chi}_R\big(\tau_R, u_R, q/\mu, \omega/D_R\mu^{2+a}\big)^{-1}$;
solve RG equation with *method of characteristics*, $\mu \to \mu\, \ell$

$$\Longrightarrow\ \chi_R(\ell) = \chi_R(1)\, \ell^{-2}\, \exp\left[- \int_1^\ell \gamma_S(\ell')\, \frac{d\ell'}{\ell'} \right] ,$$

with *running couplings*, initial values D_R, τ_R, u_R at $\ell = 1$:

$$\ell\, \frac{d\widetilde{D}(\ell)}{d\ell} = \widetilde{D}(\ell)\, \gamma_D(\ell)\ ,\ \ell\, \frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell)\, \gamma_\tau(\ell)\ ,\ \ell\, \frac{d\tilde{u}(\ell)}{d\ell} = \beta_u(\ell)\ .$$

Infrared-stable RG fixed point: $\beta_u(u^*) = 0$, $\beta_u'(u^*) > 0$;
in its vicinity: $\widetilde{D}(\ell) \approx D_R\, \ell^{\gamma_D^*}$, $\tilde{\tau}(\ell) \approx \tau_R\, \ell^{\gamma_\tau^*}$, therefore:

$$\chi_R(\tau_R, q, \omega)^{-1} \approx \mu^2 \ell^{2+\gamma_S^*}\, \hat{\chi}_R\left(\tau_R\, \ell^{\gamma_\tau^*}, u^*, \frac{q}{\mu\, \ell}, \frac{\omega}{D_R \mu^{2+a}\, \ell^{2+a+\gamma_D^*}} \right)^{-1}$$

$\Longrightarrow$ *matching* $\ell = |q|/\mu$ yields dynamic scaling form with

$$\eta = -\gamma_S^*\ , \quad \nu = -1/\gamma_\tau^*\ , \quad z = 2 + a + \gamma_D^*\ . \tag{6}$$

Explicitly in systematic $\epsilon = 4 - d$ expansion:

$$\beta_u = u_R \left[-\epsilon + \frac{n+8}{6}\, u_R + O(u_R^2) \right] \Longrightarrow\ u_H^* = \frac{6\,\epsilon}{n+8} + O(\epsilon^2)\ .$$

$d < 4$: *Heisenberg* fixed point u_H^* stable, since $\beta_u'(u_H^*) > 0$,

$$\eta = \frac{n+2}{2\,(n+8)^2}\, \epsilon^2 + O(\epsilon^3)\ , \quad \frac{1}{\nu} = 2 - \frac{n+2}{n+8}\, \epsilon + O(\epsilon^2)\ ;$$

model A: $z = 2 + c\,\eta$, with $c = 6 \ln \frac{4}{3} - 1 + O(\epsilon)$;
model B: $\gamma_D^* = -\eta \Longrightarrow$ *exact* scaling relation $z = 4 - \eta$.
$d > 4$: *Gaussian* fixed point $u_0^* = 0 \Longrightarrow$ mean-field exponents.
$d = 4$: *logarithmic* corrections: $\tilde{u}(\ell) = u_R \left(1 - \frac{n+8}{6}\, u_R\, \ln \ell \right)^{-1}$,
$\tilde{\tau}(\ell) \sim \tau_R\, \ell^{-2} (\ln |\ell|)^{-(n+2)/(n+8)} \Longrightarrow \xi \propto \tau_R^{-1/2}\, (\ln \tau_R)^{(n+2)/2(n+8)}$.

1.5 Critical dynamics with reversible mode-couplings

Microscopic variables at $T = 0$: $\partial_t S^\alpha_m(x,t) = \{H[S_m], S^\alpha_m(x,t)\}$;
Poisson brackets: $\{A, B\} \leftrightarrow \frac{i}{\hbar}[A, B]$ (correspondence principle).
Coarse-graining: $S^\alpha_m \to S^\alpha$ *mesoscopic* hydrodynamic fields;
slow modes should provide complete description: expand

$$\{\mathcal{H}[S], S^\alpha(x)\} = \int d^d x' \sum_\beta \frac{\delta \mathcal{H}[S]}{\delta S^\beta(x')} \, Q^{\beta\alpha}(x', x) \ ,$$
$$Q^{\alpha\beta}(x, x') = \{S^\alpha(x), S^\beta(x')\} = -Q^{\beta\alpha}(x', x) \ .$$

Equilibrium condition: $\mathcal{P}[S, t] \to \mathcal{P}_{\rm eq}[S]$ as $t \to \infty$ *provided* probability current is divergence-free in space spanned by $S^\alpha(x)$:

$$\int d^d x \sum_\alpha \frac{\delta}{\delta S^\alpha(x)} \left(F^\alpha_{\rm rev}[S] \; e^{-\mathcal{H}[S]/k_{\rm B}T} \right) = 0 \ , \qquad (7)$$

often more crucial than Einstein relation (2).
$\Longrightarrow$ *Reversible mode-couplings* with finite-T corrections:

$$F^\alpha_{\rm rev}[S](x) = -\int d^d x' \sum_\beta \left[Q^{\alpha\beta}(x, x') \frac{\delta \mathcal{H}[S]}{\delta S^\beta(x')} - k_{\rm B}T \, \frac{\delta Q^{\alpha\beta}(x, x')}{\delta S^\beta(x')} \right] ,$$

complete coupled set of stochastic differential equations:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = F^\alpha_{\rm rev}[S](x,t) - D^\alpha (i\nabla)^{a_\alpha} \frac{\delta \mathcal{H}[S]}{\delta S^\alpha(x,t)} + \zeta^\alpha(x,t) \ . \quad (8)$$

Example: *Heisenberg model* for *isotropic ferromagnets*:

$$H[\{\vec{S}_j\}] = -\frac{1}{2} \sum_{j,k=1}^{N} J_{jk} \, \vec{S}_j \cdot \vec{S}_k \ , \quad \left[S^\alpha_j, S^\beta_k \right] = i\hbar \sum_\gamma \epsilon^{\alpha\beta\gamma} S^\gamma_j \, \delta_{jk} \ .$$

$\Longrightarrow$ Poisson brackets $Q^{\alpha\beta}(x, x') = -g \sum_\gamma \epsilon^{\alpha\beta\gamma} S^\gamma(x) \, \delta(x - x')$.
Model J Langevin equations, with $\langle \vec{\zeta}(x,t) \rangle = 0$:

$$\frac{\partial \vec{S}(x,t)}{\partial t} = -g \, \vec{S}(x,t) \times \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + D \nabla^2 \, \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + \vec{\zeta}(x,t) \, ,$$
$$\left\langle \zeta^\alpha(x,t) \, \zeta^\beta(x',t') \right\rangle = -2D \, k_{\rm B}T \, \nabla^2 \delta(x - x') \delta(t - t') \delta^{\alpha\beta} \ .$$

$\Longrightarrow$ Additional *mode-coupling vertex* in action, see Fig. 5(a):

$$\mathcal{A}_J[\widetilde{S}, S] = -g \int d^d x \int dt \sum_{\alpha,\beta,\gamma} \epsilon^{\alpha\beta\gamma} \widetilde{S}^\alpha S^\beta \left(\nabla^2 S^\gamma + h^\gamma\right) \ ;$$

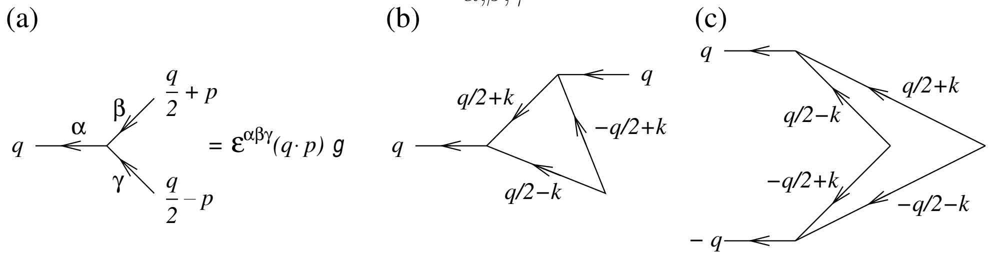

Figure 5: (a) Mode-coupling three-point vertex for model J. One-loop Feynman diagrams for the propagator (b) and noise vertex (c) renormalizations in model J. The same graphs apply for the DDS and KPZ Langevin equations.

$[g] = \mu^{3-d/2}$, expect *dynamical* upper critical dimension $d_c' = 6$. Exploit that the S^α are the generators of the rotation group:

$$\langle S^\alpha(x,t)\rangle_h = g \int_0^t dt' \sum_\beta \epsilon^{\alpha\beta\gamma} \langle S^\beta(x,t')\rangle_h h^\gamma(t) \ ,$$

non-linear susceptibility $R^{\alpha;\beta\gamma} = \delta^2 \langle S^\alpha \rangle / \delta h^\beta \, \delta h^\gamma |_{h=0}$ therefore:

$$\int d^d x' \, R^{\alpha;\beta\gamma}(x,t; x - x', t - t') = g \, \epsilon^{\alpha\beta\gamma} \, \chi^{\beta\beta}(x,t) \, \Theta(t) \, \Theta(t - t') \ .$$

Must hold after renormalization; with $B_d = \Gamma(4 - d/2)/2^d \, d \, \pi^{d/2}$:

$$g_R^2 = Z_g \, g^2 \, B_d \, \mu^{d-6} \implies Z_g = Z_S \ ,$$
$$f = g^2/D^2 \ : \quad \beta_f = \mu \, \partial_\mu |_0 f_R = f_R \, (d - 6 + \gamma_S - 2\,\gamma_D) \ .$$

Thus, at any *non-trivial* RG fixed point $0 < f^* < \infty$, we have the *exact* scaling relation (to *all* orders in perturbation theory):

$$d < 6 \ : \quad z = 4 + \gamma_D^* = 4 + \frac{d - 6 + \gamma_S^*}{2} = \frac{d + 2 - \eta}{2} \ .$$

Explicit one-loop calculation, Fig. 5(b), with $\varepsilon = 6 - d$, yields:

$$\gamma_D = -f_R + \mathcal{O}(u_R^2, f_R^2) \ , \quad f_J^* = \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2) \ .$$

Scaling functions: self-consistent one-loop/*mode-coupling* theory.

1.6 Critical relaxation, initial slip, and ageing

Rapid *quench* from disordered state at $T \gg T_c$ to $T \approx T_c$: relaxation from Gaussian random initial conditions:

$$e^{-\mathcal{H}_0[S]} = \exp\left(-\frac{\Delta}{2} \int d^d x \sum_\alpha \left[S^\alpha(x,0) - a^\alpha(x) \right]^2 \right) \ ,$$

$[\Delta] = \mu^2 \Longrightarrow \Delta \to \infty$ under RG, *Dirichlet boundary conditions*; *propagators*: $G_0(q,;t-t') = e^{-Dq^a\,(r+q^2)\,(t-t')}\,\Theta(t-t')$, $C_D(q;t,t') = (r+q^2)^{-1} \left(e^{-Dq^a\,(r+q^2)\,|t-t'|} - e^{-Dq^a\,(r+q^2)\,(t+t')} \right)$, *temporal* translation invariance broken by initial state; equilibration time $t_c \sim |\tau|^{-z\nu} \to \infty$ as $\tau \to 0$: critical *'ageing'*. Models A / B: follow theory of *boundary critical phenomena*, additional singularities on 'surface' $t + t' = 0$ only for model A:

$$\widetilde{S}^\alpha_R(x,0) = (Z_0 Z_{\widetilde{S}})^{1/2}\, \widetilde{S}^\alpha(x,0) \ , \quad Z_0 = 1 - \frac{n+2}{6}\,\frac{u_R}{\epsilon} \ ,$$

$$a = 0 \ : \ \theta = \frac{\gamma_0^*}{2\,z} = \frac{n+2}{4\,(n+8)}\,\epsilon + O(\epsilon^2) \ , \quad a = 2 \ : \ \theta = 0 \ .$$

Employ *short-distance operator product expansion*:

$$t \to 0 \ : \quad \widetilde{S}(x,t) = \tilde{\sigma}(t)\, \widetilde{S}_0(x) \ , \quad S(x,t) = \sigma(t)\, \widetilde{S}_0(x) \ .$$

Analysis yields for *short-time* scaling limit $t'/t \to 0$:

$$\chi(q;t,t' \to 0) = |q|^{z-2+\eta} \left(\frac{t}{t'} \right)^\theta \hat{\chi}_0(|q|\xi, |q|^z Dt) \ ,$$

$$C(q;t,t' \to 0) = q^{-2+\eta} \left(\frac{t}{t'} \right)^{\theta-1} \hat{C}_0(|q|\xi, |q|^z Dt) \ ,$$

$$\langle S(t) \rangle = S_0\, t^{\theta'}\, \hat{S}\big(S_0\, t^{\theta' + \beta/z\nu} \big) \ , \quad C(0;t,t' \to 0) \sim t^{\theta' - d/z}$$

$$a = 0 \ : \ \theta' = \theta - \frac{z - 2 + \eta}{z} \ , \quad a = 2 \ : \ \theta' = \theta = 0 \ .$$

Universal *fluctuation–dissipation* ratios ($\Longrightarrow$ *A. Gambassi*). Model J: $\theta = (z - 4 + \eta)/z = -(6 - d - \eta)/(d + 2 - \eta)$; mode-coupling, *non-conserved* order parameter: θ *not* universal.

1.7 Nonequilibrium relaxational critical dynamics

What happens if detailed balance conditions are violated?

Change noise strength $D \to \widetilde{D}$ in models A / B, violates (2): can obviously be absorbed into rescaled effective temperature, $k_{\rm B}T \to k_{\rm B}T' = \widetilde{D}/D$; formally with action (3):

$$\mathcal{A}[\tilde{S}, S] = \int d^d x \int dt \sum_\alpha \tilde{S}^\alpha \Big[\partial_t \, S^\alpha + D(i\nabla)^a \left(r - \nabla^2 \right) S^\alpha - \widetilde{D} \, (i\nabla)^a \, \tilde{S}^\alpha + D \, \frac{u}{6} \, (i\nabla)^a \, S^\alpha \sum_\beta S^\beta S^\beta \Big] \ .$$

Rescaling $\tilde{S}^\alpha \to \tilde{S}'^\alpha = \tilde{S}^\alpha \sqrt{\widetilde{D}/D}$, $S^\alpha \to S'^\alpha = S^\alpha \sqrt{D/\widetilde{D}}$
yields equilibrium form, with $u \to \tilde{u} = u \, \widetilde{D}/D$;
for model J similarly: $g \to \tilde{g} = g \sqrt{\widetilde{D}/D}$.

$\Longrightarrow$ *No* effect on *universal* properties near u_H^*, f_J^*.

Models with *conserved order parameter*:

allow for anisotropic relaxation $-D \, \nabla^2 \to -D_\parallel \, \nabla_\parallel^2 - D_\perp \, \nabla_\perp^2$ *and* noise correlations $-\widetilde{D} \, \nabla^2 \to -\widetilde{D}_\parallel \, \nabla_\parallel^2 - \widetilde{D}_\perp \, \nabla_\perp^2$;
$\widetilde{D}_\parallel / D_\parallel \neq \widetilde{D}_\perp / D_\perp$: longitudinal and transverse spatial sectors coupled to heat baths with temperatures $T_\parallel > T_\perp$ (say).
Transverse sector softens first, longitudinal sector non-critical $\Longrightarrow$ *anisotropic* scaling: $[q_\perp] = \mu$, $[q_\parallel] = \mu^2$, $[\omega] = \mu^4$, hence $[\widetilde{D}_\perp] = [D_\perp] = \mu^0$, $[\widetilde{D}_\parallel] = [D_\parallel] = \mu^{-2}$; leads to randomly driven *two-temperature model B* ($D = D_\perp, c = r_\parallel D_\parallel / D_\perp$):

$$\frac{\partial S^\alpha(x,t)}{\partial t} = D \left[c \nabla_\parallel^2 + \nabla_\perp^2 (r - \nabla_\perp^2) \right] S^\alpha(x,t) + D \nabla_\perp^2 \, \frac{\tilde{u}}{6} \, S^\alpha(x,t) \sum_\beta [S^\beta(x,t)]^2 + \zeta^\alpha(x,t) \ ,$$

$$\left\langle \zeta^\alpha(x,t) \zeta^\beta(x',t') \right\rangle = -2D \, \nabla_\perp^2 \, \delta(x - x') \delta(t - t') \delta^{\alpha\beta} \ .$$

Equivalent to *equilibrium* relaxational dynamics

$$\frac{\partial S^\alpha(x,t)}{\partial t} = D\nabla_\perp^2 \, \frac{\delta \mathcal{H}_{\rm eff}[S]}{\delta S^\alpha(x,t)} + \zeta^\alpha(x,t) \ ,$$

derived from an effective *long-range* Hamiltonian

$$\mathcal{H}_{\rm eff}[S] = \int_q \sum_\alpha \frac{c q_\parallel^2 + q_\perp^2 (r_\perp + q_\perp^2)}{2\, q_\perp^2} \, S^\alpha(q)\, S^\alpha(-q) + \frac{\tilde{u}}{4!} \int d^d x \sum_{\alpha,\beta} [S^\alpha(x)]^2 [S^\beta(x)]^2 \ .$$

Power counting: $[\tilde{u}] = \mu^{4-d_\parallel-d}$, hence $d_c = 4 - d_\parallel$.
Scaling law for, e.g., dynamic response function:

$$\chi(\tau_\perp, q_\parallel, q_\perp, \omega) = q_\perp^{-2+\eta} \, \hat{\chi}\left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{\sqrt{c}\, q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{D q_\perp^z} \right) \ .$$

$\Longrightarrow$ One new *anisotropy* exponent; *exact* scaling relations:

$$z = 4 - \eta \ , \qquad \Delta = 1 - \frac{\gamma_c^*}{2} = 1 - \frac{\eta}{2} = \frac{z}{2} - 1 \ ,$$
$$z_\parallel = \frac{z}{1+\Delta} = 2 \ , \quad \nu_\parallel = \nu\, (1+\Delta) = \frac{\nu}{2}\, (4-\eta) \ ,$$

since to *all* orders in the perturbation expansion:

$$\Gamma^{(1,1)}(q_\parallel, q_\perp = 0, \omega) = i\omega + D\, c\, q_\parallel^2 \ \Longrightarrow \ Z_c = Z_D^{-1} = Z_S^{-1} \ .$$

Explicitly to one-loop order (unaltered combinatorics):

$$\epsilon = 4 - d - d_\parallel > 0 \ : \quad \eta = 0 + O(\epsilon^2) \ , \quad \frac{1}{\nu} = 2 - \frac{n+2}{n+8}\, \epsilon + O(\epsilon^2) \ .$$

Similarly constructed non-equilibrium *two-temperature model J*: *not* equivalent to equilibrium system: condition (7) not satisfied; RG runaway flow, no fixed point (?); interpretation unclear.

1.8 Driven diffusive systems

Consider *conserved* particle density, driven along '$\|$' direction; *continuity equation*: $\partial_t S(x,t) + \nabla \cdot J(x,t) = 0$, $\langle S(x,t) \rangle = 0$.
Transverse sector, $d_\perp = d-1$: $J_\perp(x,t) = -D\nabla_\perp S(x,t) + \eta(x,t)$, current along the direction of an external drive:
$J_\|(x,t) = \langle J_\| \rangle - D\, c\, \nabla_\| S(x,t) - \frac{1}{2} Dg\, S(x,t)^2 + \zeta(x,t)$,
with $\langle \eta_i \rangle = 0 = \langle \zeta \rangle$ and noise correlations:

$$\langle \eta_i(x,t)\, \eta_j(x',t') \rangle = 2D\, \delta_{ij} \delta(x-x') \delta(t-t') \ ,$$
$$\langle \zeta(x,t)\, \zeta(x',t') \rangle = 2D\, \tilde{c}\, \delta(x-x') \delta(t-t') \ .$$

Janssen–De Dominicis response functional (3) for *DDS*:

$$\mathcal{A}[\tilde{S}, S] = \int d^d x \int dt\ \tilde{S} \Big[\partial_t S - D \left(\nabla_\perp^2 + c\, \nabla_\|^2 \right) S + D \left(\nabla_\perp^2 + \tilde{c}\, \nabla_\|^2 \right) \tilde{S} - \frac{D\, g}{2}\, \nabla_\| S^2 \Big] \ ;$$

'massless' theory, *generically* scale-invariant.
Vertex $\sim iq_\| \Longrightarrow Z_{\tilde{S}} = Z_S = Z_D = 1$, thus $\eta = 0$, $z = 2$.
Galilean transformation: $S'(x'_\perp, x'_\|, t') = S(x_\perp, x_\| - Dgv\, t, t) - v$, leaves Langevin equation / action *invariant*, $v \sim S \Longrightarrow Z_g = 1$.
Explicit one-loop calculation (Fig. 5), $C_d = \Gamma(2-d/2)/2^{d-1}\pi^{d/2}$:

$$w = \tilde{c}/c \ , \quad v = g^2\, c^{-3/2} \ , \quad v_R = Z_c^{3/2}\, v\, C_d\, \mu^{d-2} \ , \quad d_c = 2 \ ;$$
$$\gamma_c = -\frac{v_R}{16} (3 + w_R) \ , \quad \gamma_{\tilde{c}} = -\frac{v_R}{32} \left(3w_R^{-1} + 2 + 3w_R \right) \ .$$
$$\Longrightarrow \ \beta_w = w_R\, (\gamma_{\tilde{c}} - \gamma_c) = -\frac{v_R}{32} (w_R - 1)(w_R - 3) \ ,$$
$$\beta_v = v_R \left(d - 2 - \frac{3}{2}\, \gamma_c \right) \ .$$

At *any* non-trivial RG fixed point $0 < v^* < \infty$: $w_R^* = 1$ *stable*;

$$d < 2 \ : \quad \Delta = -\frac{\gamma_c^*}{2} = \frac{2-d}{3} \ , \quad z_\| = \frac{2}{1+\Delta} = \frac{6}{5-d} \ .$$

Noisy Burgers equation ($Dg = 1$, simplified fluid dynamics):

$$\frac{\partial u(x,t)}{\partial t} + \frac{Dg}{2} \nabla \left[u(x,t)^2 \right] = D \, \nabla^2 u(x,t) + \zeta(x,t) \ ,$$
$$\langle \zeta_i \rangle = 0 \, , \ \langle \zeta_i(x,t) \, \zeta_j(x',t') \rangle = -2D \, \nabla_i \nabla_j \, \delta(x-x') \delta(t-t') \ .$$

$d = 1$: *identical* with DDS Langevin equation, so $z = 3/2$.
Equilibrium condition (7) with $\mathcal{P}_{\rm eq}[u] \propto \exp \left[-\frac{1}{2} \int u(x)^2 \, d^dx \right]$:

$$\int d^dx \, \frac{\delta}{\delta u(x,t)} \cdot \left[\nabla u(x,t)^2 \right] e^{-\frac{1}{2} \int u(x',t)^2 \, d^dx'}$$
$$= \int \left[2 \nabla \cdot u(x,t) - u(x,t) \cdot \nabla u(x,t)^2 \right] d^dx \, e^{-\frac{1}{2} \int u(x',t)^2 \, d^dx'} \ ,$$

first term vanishes, second one *only* in one dimension:
$- \int u (du^2/dx) dx = \int u^2 (du/dx) dx = \frac{1}{3} \int (du^3/dx) dx = 0$;
$\Longrightarrow$ *'hidden' fluctuation-dissipation theorem* in $d = 1$.

Driven model B, critical DDS: conserved scalar field S subject to phase transition, but only *transverse* sector critical:

$$\mathcal{A}[\widetilde{S}, S] = \int d^dx \int dt \ \widetilde{S} \Big[\partial_t \, S - D \nabla_\perp^2 \left(r - \nabla_\perp^2 \right) S - Dc \, \nabla_\parallel^2 \, S$$
$$+ D \left(\nabla_\perp^2 \, \widetilde{S} - \frac{g}{2} \, \nabla_\parallel \, S^2 - \frac{u}{6} \, \nabla_\perp^2 \, S^3 \right) \Big] \ .$$

$[g^2] = \mu^{5-d}$, so $d_c = 5$, and $[u] = \mu^{3-d}$ (dangerously) *irrelevant*; remaining vertex $\sim i q_\parallel \Longrightarrow Z_{\widetilde{S}} = Z_S = Z_D = 1$, thus *exactly*:

$$\eta = 0 \ , \quad \nu = 1/2 \ , \quad z = 4 \ .$$

Galilean invariance as before: $Z_g = 1$, therefore:

$$v = g^2 \, c^{-3/2} \ , \quad \beta_v = v_R \left(d - 5 - \frac{3}{2} \, \gamma_c \right) \ .$$

$\Longrightarrow$ scaling exponents to *all* orders in perturbation theory:

$$d < 5 \ : \quad \Delta = 1 - \frac{\gamma_c^*}{2} = \frac{8-d}{3} \ , \quad z_\parallel = \frac{4}{1+\Delta} = \frac{12}{11-d} \ .$$

DDS: non-zero *three-point* correlations (*not* with random drive).

1.9 Driven interfaces, KPZ roughening transition

Ising *domain wall* $z = h(x,t)$, *driven* by external magnetic field: interfacial energy $\mathcal{H}[h] = \int \left[\sqrt{1 + (\nabla h)^2} - g\, h\right] d^d x$.

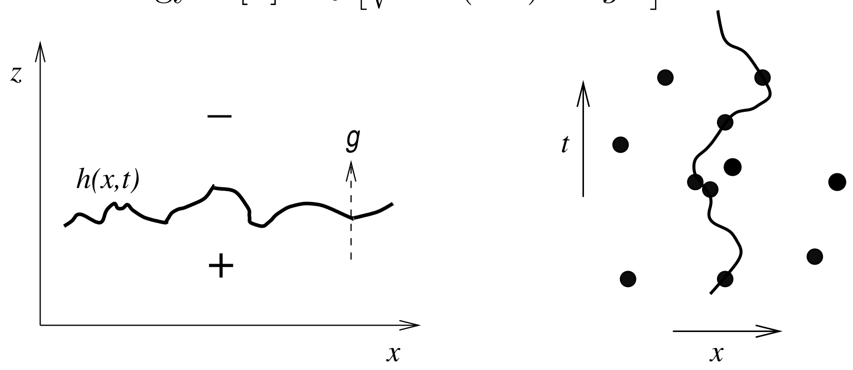

Figure 6: Left: Ising interface, characterized by a height function $z = h(x,t)$, driven along the z direction by a uniform magnetic field g; right: directed line interacting with randomly distributed point defects.

Assume *relaxational* kinetics, rate $D\sqrt{1 + (\nabla h)^2}$; for $(\nabla h)^2 \ll 1$:

$$\frac{\partial h(x,t)}{\partial t} = -D\sqrt{1 + (\nabla h)^2}\, \frac{\delta \mathcal{H}[h]}{\delta h(x,t)} + \zeta(x,t)$$
$$\approx D\, \nabla^2 h(x,t) + Dg + \frac{Dg}{2} \left[\nabla h(x,t)\right]^2 + \zeta(x,t) \ ;$$

$S(x,t) = h(x,t) - Dgt \Longrightarrow$ *Kardar-Parisi-Zhang* equation:

$$\frac{\partial S(x,t)}{\partial t} = D\, \nabla^2 S(x,t) + \frac{Dg}{2} \left[\nabla S(x,t)\right]^2 + \zeta(x,t) \ ,$$
$$\langle \zeta \rangle = 0 \ , \quad \langle \zeta(x,t)\, \zeta(x',t') \rangle = 2D\, \delta(x - x')\delta(t - t') \ .$$

Correlation function, *roughness* exponent:

$$C(x,t) = |x|^{2\chi}\, \hat{C}(t/|x|^z) \ , \quad \chi = (2 - d - \eta)/2 \ .$$

$g = 0$: *Edwards-Wilkinson* model, $\eta = 0$, $\chi = 1 - \frac{d}{2}$, $z = 2$.
Power counting: $[g] = \mu^{(2-d)/2}$, critical dimension $d_c = 2$.
$u(x,t) = -\nabla\, S(x,t) \Longrightarrow$ noisy Burgers equation; consequently in $d = 1$: $\mathcal{P}_{\rm eq}[S] \propto \exp\left[-\frac{1}{2} \int (\nabla S)^2\, dx\right]$, $\eta = 0$, $\chi = \frac{1}{2}$, $z = \frac{3}{2}$.

Renormalization, exploiting that $\Gamma^{(1,1)}(q=0,\omega) = i\omega$:

$$S_R = Z_S^{1/2} S \, , \ \widetilde{S}_R = Z_S^{-1/2} \widetilde{S} \, , \ D_R = Z_D D \, , \ g_R^2 = Z_g \, g^2 C_d \, \mu^{d-2} \ .$$

Galilean transformation translates to interface *tilt invariance*:

$$\begin{aligned} & S'(x',t') = S(x - Dgv\,t, t) - v \cdot x \\ \Longrightarrow \quad & Z_D (Z_g Z_S)^{1/2} = 1 \ , \quad \beta_g = g_R^2 \, (d - 2 - \gamma_S - 2\gamma_D) \ . \end{aligned}$$

So once again, at *any* non-trivial RG fixed point $0 < (g^*)^2 < \infty$:

$$\chi + z = \frac{2 - d + \gamma_S^*}{2} + \gamma_D^* + 2 = 2 \ .$$

Explicitly to one-loop order at *fixed* dimension:

$$\gamma_S = \frac{d-1}{2d} \, g_R^2 \ , \quad \gamma_D = \frac{d-2}{4d} \, g_R^2 \ , \quad (g^*)^2 = \frac{2d\,(d-2)}{2d-3} \ ;$$

recovers exactly known exponents in $d = 1$.

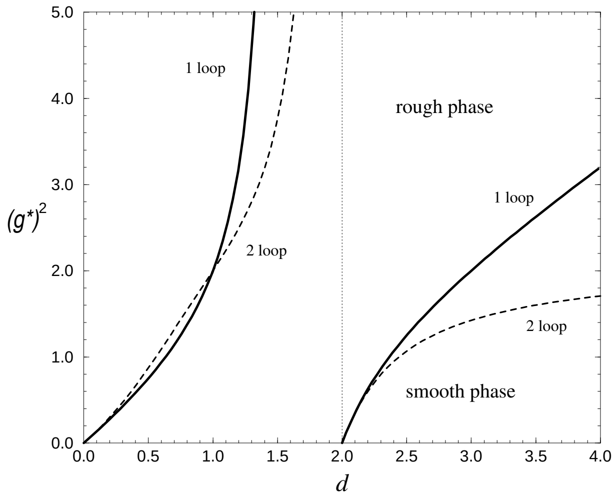

Figure 7: KPZ fixed point $(g^*)^2$ as function of dimensions d in one-loop (solid line) and two-loop approximation (dashed). For $d < 2$, $g^* > 0$ is stable. For $d \geq 2$, it becomes unstable, and controls a continuous roughening transition.

Two-loop calculation: $(g^*)^2$ diverges only at $d_c = 2$, see Fig. 7.

$d > 2$: non-equilibrium *roughening transition* separates smooth EW ($g \to 0$, $\chi < 0$) and strong-coupling rough ($g \to \infty$) phases.
Partition function for directed polymer subject to point disorder:

$$Z(x,t) = \int_{y(0)=0}^{y(t)=x} \mathcal{D}[y] \exp\left\{ -\frac{1}{k_{\rm B}T} \int_0^t \left[\frac{\varepsilon}{2} \left(\frac{dy}{ds} \right)^2 + V(y(s),s) \right] ds \right\} ,$$

equivalent to 'imaginary-time' Schrödinger equation:

$$k_{\rm B}T \, \frac{\partial Z(x,t)}{\partial t} = \frac{(k_{\rm B}T)^2}{2\varepsilon} \nabla^2 Z(x,t) + V(x,t)\, Z(x,t) \ ;$$

Cole-Hopf transform $S(x,t) = (k_{\rm B}T/Dg\varepsilon) \ln Z(x,t) \Longrightarrow$ KPZ.
Critical exponents: expand about *lower* critical dimension d_c;

$$n(x,t) = \frac{2}{g} \exp\left[\frac{g}{2} S(x,t) \right] , \ \tilde{n}(x,t) = \tilde{S}(x,t) \exp\left[-\frac{g}{2} S(x,t) \right]$$

$$\Longrightarrow \ \mathcal{A}[n,\tilde{n}] = \int d^dx \int dt \left[\tilde{n} \left(\partial_t - D\, \nabla^2 \right) n - \frac{Dg^2}{4} \left(\tilde{n}\, n \right)^2 \right] \ .$$

No propagator renormalization $\Longrightarrow$ $z_c = 2$, $\chi_c = 0$, $\eta_c = d - 2$,
vertex renormalization: to *all* orders via geometric series:

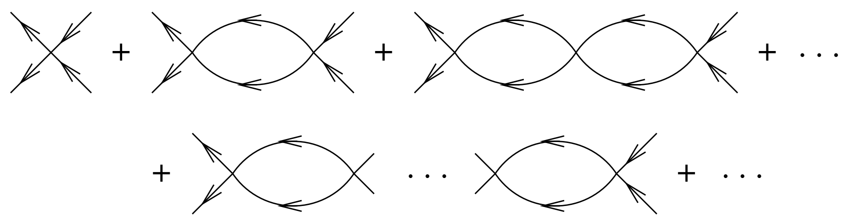

Figure 8: Vertex renormalization for KPZ and the reaction $A + A \to \emptyset$.

$$Z_g^{-1} = 1 + \frac{g^2\, C_d \mu^{d-2}}{4(d-2)} \ , \quad \beta_g = g_R^2 \left(d - 2 - \frac{g_R^2}{4} \right) \ ;$$

define diverging *crossover* length scale:

$$\xi \propto |g_c - g_R|^{-\nu_c} \ , \quad g_c^2 = 4\,(d-2) \ , \quad \nu_c^{-1} = -\beta_g'^{\,*} = d - 2 \ .$$

No successful analytic approach to non-perturbative regime:
$(\nabla S)^2$ *not* small; universality ? continuum description valid ?

2 Reaction–diffusion systems

2.1 Introduction: reactions, population dynamics

'Particles' $A, B, \ldots$ hop to nearest neighbors, upon encounter: species change, annihilate, proliferate, ... $\Longrightarrow$ *diffusion-limited*; assume homogeneous mixing $\Longrightarrow$ *mean-field rate* equations.

Annihilation $k\,A \to l\,A$ ($l < k$): $\partial_t\, a(t) = -(k-l)\,\lambda\, a(t)^k$;

$$k = 1 \ : \ \ a(t) = a(0)\, e^{-\lambda t} \ ,$$

$$k \geq 2 \ : \ \ a(t) = \left[a(0)^{1-k} + (k-l)(k-1)\,\lambda\, t\right]^{-1/(k-1)} \ ;$$

expect *depletion zones* for $d \leq d_c \Longrightarrow$ *slower* decay power laws.
Two-species pair annihilation $A + B \to \emptyset$ (without mixing): particle species *segregate* for $d \leq d_s \Longrightarrow$ *reaction fronts.*
Competing $A \to \emptyset$, $A \rightleftharpoons A + A$: $\partial_t\, a(t) = (\sigma - \kappa)\, a(t) - \lambda\, a(t)^2$;
$\Longrightarrow$ *continuous* phase transition at $\sigma_c = \kappa$:

$\sigma > \kappa \ : \ a(t) \to a_\infty = (\sigma - \kappa)/\lambda$ *active* phase;
$\sigma < \kappa \ : \ a(t) \to 0$ *inactive, absorbing* state;
$\sigma = \kappa \ : \ a(t) \sim (\lambda\, t)^{-1}$ *critical* power law.

Include fluctuations: critical exponents, *universality classes* ?

Population dynamics: same mathematical framework;
e.g., *Lotka–Volterra* predator–prey competition (1920, 1926):
death $A \to \emptyset$, *offspring* $B \to B + B$, *predation* $A + B \to A + A$:

$$\partial_t\, a(t) = \lambda\, a(t)\, b(t) - \kappa\, a(t) \ , \quad \partial_t\, b(t) = \sigma\, b(t) - \lambda\, a(t)\, b(t) \ .$$

$K(t) = \lambda[a(t) + b(t)] - \sigma \ln a(t) - \kappa \ln b(t) = \text{const.} \ \Longrightarrow$
regular population *oscillations*, determined by *initial* state.
Include spatial degrees of freedom (diffusion), stochasticity:
'pursuit and evasion' waves in coexistence phase $\Longrightarrow$
complex dynamical patterns, *erratic* oscillations (*finite* systems);
predator *extinction threshold* (absorbing state).

2.2 Field theory representation of master equations

Master equation for probability $P(\{n_i\};t)$, $n_i = 0, 1, 2, \ldots$ provides *balance* of gain and loss terms; e.g., $A + A \to \emptyset, A$:

$$\begin{aligned}\partial_t\, P(n_i;t) = \lambda\,(n_i+2)\,(n_i+1)\,P(\ldots,n_i+2,\ldots;t) \\ +\lambda'\,(n_i+1)\,n_i\,P(\ldots,n_i+1,\ldots;t) \\ -(\lambda+\lambda')\,n_i\,(n_i-1)\,P(\ldots,n_i,\ldots;t)\ ,\end{aligned}$$

with initial *Poisson* distribution $P(\{n\},0) = \prod_i (\bar{n}_0^{n_i}\, e^{-\bar{n}_0}/n_i!)$. Introduce second-quantized *bosonic operator* representation:

$$\begin{aligned}&[a_i, a_j] = 0\,,\ \ [a_i, a_j^\dagger] = \delta_{ij}\,,\ \ a_i|0\rangle = 0\ ,\ \ a_i\,|n_i\rangle = n_i\,|n_i-1\rangle\,, \\ &a_i^\dagger\,|n_i\rangle = |n_i+1\rangle \quad \Longrightarrow\ \ |\{n_i\}\rangle = \prod_i \left(a_i^\dagger\right)^{n_i}\,|0\rangle\ .\end{aligned}$$

Time evolution of *state vector* $|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\};t)\,|\{n_i\}\rangle$:

$$\partial_t\,|\Phi(t)\rangle = -H\,|\Phi(t)\rangle\ , \quad H = \sum_i H_i(a_i^\dagger, a_i)\ ;$$

e.g., diffusion-limited annihilation and coagulation:

$$H = D\sum_{<ij>}(a_i^\dagger - a_j^\dagger)(a_i - a_j) - \sum_i \left[\lambda(1-a_i^{\dagger 2})\,a_i^2 + \lambda'(1-a_i^\dagger)\,a_i^\dagger a_i^2\right];$$

note: first term $\leftrightarrow$ physical *process*, second term: *'order'*.
Akin non-Hermitian imaginary-time Schrödinger equation, but need *projection* $\langle\mathcal{P}| = \langle 0|\prod_i e^{a_i}$, $\langle\mathcal{P}|0\rangle = 1$ for *statistical averages*

$$\langle F(t)\rangle = \sum_{\{n_i\}} F(\{n_i\})\,P(\{n_i\};t) = \langle\mathcal{P}|\,F(\{a_i\})\,|\Phi(t)\rangle\ ;$$

probability conservation:

$$1 = \langle\mathcal{P}|\Phi(t)\rangle = \langle\mathcal{P}|e^{-H\,t}|\Phi(0)\rangle\ \Longrightarrow\ \langle\mathcal{P}|H = 0\ .$$

$\left[e^a, a^\dagger\right] = e^a \Longrightarrow$ commuting $e^{\sum_i a_i}$ with H shifts $a_i^\dagger \to 1 + a_i^\dagger$;
$\Longrightarrow H_i(a_i^\dagger \to 1, a_i) = 0$, in averages replace $a_i^\dagger a_i \to a_i$, i.e., *density* $a(t) = \langle a_i\rangle$, two-point operator $a_i^\dagger a_i\, a_j^\dagger a_j \to a_i\delta_{ij} + a_i a_j$.

Construct *path integral* representation via *coherent states*:

$$a_i\,|\phi_i\rangle = \phi_i\,|\phi_i\rangle \ , \quad |\phi_i\rangle = \exp\left(-\frac{1}{2}\,|\phi_i|^2 + \phi_i\,a_i^\dagger\right)|0\rangle \ ,$$

$$\Longrightarrow \ 1 = \int \prod_i \frac{d^2\phi_i}{\pi}\,|\{\phi_i\}\rangle\,\langle\{\phi_i\}| \quad \text{(overcomplete)} \ .$$

Split time evolution into infinitesimal steps, standard procedures

$$\Longrightarrow \ \langle F(t)\rangle \propto \int \prod_i \mathcal{D}[\phi_i]\,\mathcal{D}[\phi_i^*]\,F(\{\phi_i\})\,e^{-\mathcal{A}[\phi_i^*,\phi_i]} \ ,$$

$$\mathcal{A}[\phi_i^*,\phi_i] = \sum_i \Big(-\phi_i(t_f) + \int_0^{t_f} dt\,[\phi_i^*\,\partial_t\,\phi_i + H(\{\phi_i^*\},\{\phi_i\})] \\ -\bar{n}_0\,\phi_i^*(0)\Big) \ .$$

Continuum limit $\Longrightarrow \phi_i(t) \to \psi(x,t)$, $\phi_i^*(t) \to \hat{\psi}(x,t)$; 'bulk':

$$\mathcal{A}[\hat{\psi},\psi] = \int d^dx \int_0^{t_f} dt \left[\hat{\psi}\left(\partial_t - D\,\nabla^2\right)\psi + \mathcal{H}_r(\{\hat{\psi}\},\{\psi\})\right] \ , \quad (9)$$

microscopic stochastic field theory, *no* assumptions on noise ! $\Longrightarrow$ basis for coarse-graining, renormalization group analysis.

Pair annihilation and coagulation $A + A \to \emptyset, A$:

$$\mathcal{H}_r(\{\hat{\psi}\},\{\psi\}) = -\lambda\left(1 - \hat{\psi}^2\right)\psi^2 - \lambda'\left(1 - \hat{\psi}\right)\hat{\psi}\,\psi^2 \ ,$$

classical field equations: $\delta\mathcal{A}/\delta\psi = 0 = \delta\mathcal{A}/\delta\hat{\psi} \Longrightarrow \hat{\psi} = 1$ and

$$\partial_t\,\psi(x,t) = D\,\nabla^2\,\psi(x,t) - (2\lambda + \lambda')\,\psi(x,t)^2 \ ;$$

shift about mean-field solution $\hat{\psi}(x,t) = 1 + \tilde{\psi}(x,t)$:

$$\mathcal{H}_r(\{\tilde{\psi}\},\{\psi\}) = (2\lambda + \lambda')\,\tilde{\psi}\,\psi^2 + (\lambda + \lambda')\,\tilde{\psi}^2\,\psi^2$$

$\Longrightarrow$ aside from amplitudes, expect *identical* scaling behavior; formally equivalent to 'Langevin' equation with noise correlator $L[\psi] = -(\lambda + \lambda')\,\psi^2 < 0 \ \Longrightarrow$ *'imaginary'* multiplicative noise.

2.3 Diffusion-limited annihilation $kA \to \emptyset$

Field theory action (9) for $k\,A \to \emptyset$:

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \left[\hat{\psi} \left(\partial_t - D \nabla^2 \right) \psi - \lambda \left(1 - \hat{\psi}^k \right) \psi^k \right] \ ,$$

for $k \geq 3$ no (obvious) equivalent Langevin description.
$[\lambda] = \mu^{2-(k-1)d} \Longrightarrow d_c(k) = 2/(k-1)$, mean-field for $k > 3$.
No propagator renormalization, massless $\Longrightarrow \eta = 0$, $z = 2$;
geometric series for vertex renormalization (Bethe–Salpeter):

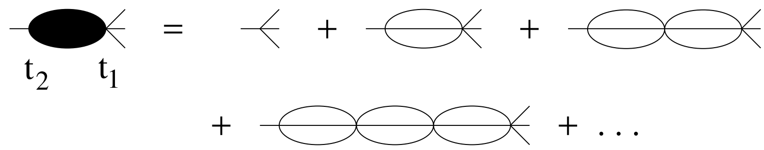

Figure 9: Vertex renormalization for diffusion-limited annihilation $3\,A \to \emptyset$.

renormalized rate: $g_R = Z_g\,(\lambda/D)\,B_{kd}\,\mu^{-2(1-d/d_c)}$:

$$Z_g^{-1} = 1 + \frac{\lambda\,B_{kd}\,\mu^{-2(1-d/d_c)}}{D\,(d_c - d)} \ , \quad B_{kd} = \frac{k!\,\Gamma(2 - d/d_c)\,d_c}{k^{d/2}\,(4\pi)^{d/d_c}} \ ,$$

$$\Longrightarrow \ \beta_g = \mu\,\partial_\mu|_0\,g_R = -\frac{2\,g_R}{d_c}\,(d - d_c + g_R) \ , \quad g^* = d_c - d \ .$$

RG equation (5) for particle *density* $a(t)$, $[a] = \mu^d$, $(\mu\ell)^2 = 1/Dt$:

$$\left[d + 2Dt\,\frac{\partial}{\partial(D\,t)} - d\,n_0\,\frac{\partial}{\partial n_0} + \beta_g\,\frac{\partial}{\partial g_R} \right] a(\mu, D, n_0, g_R, t) = 0 \ ;$$

solution: $a(\mu, D, n_0, g_R, t) \ = \ (D\mu^2\,t)^{-d/2}\,\hat{a}\Big(n_0\,(D\mu^2\,t)^{d/2}, \tilde{g}(t)\Big)$,
need to establish that result finite to *all* orders in $n_0 \to \infty$!

$$\begin{aligned}
\Longrightarrow \ k = 2 : \quad & d < 2 \ : \ a(t) \sim (D\,t)^{-d/2} \ , \\
& d = 2 \ : \ a(t) \sim (D\,t)^{-1}\ln(D\,t) \ , \\
& d > 2 \ : \ a(t) \sim (\lambda\,t)^{-1} \ ; \\
k = 3 : \quad & d = 1 \ : \ a(t) \sim \left[(D\,t)^{-1}\ln(D\,t)\right]^{1/2} \ , \\
& d > 1 : \ a(t) \sim (\lambda\,t)^{-1/2} \ .
\end{aligned}$$

2.4 Segregation and reaction fronts for $A + B \to \emptyset$

Two *distinct* species A, B, *no* reactions between same species:
mean-field rate equations: $\partial_t \, a(t) = -\lambda \, a(t) \, b(t) = \partial_t \, b(t)$.

Equal initial densities $a(0) = b(0)$: $a(t) = b(t) \sim 1/\lambda \, t$;
unequal initial densities $a(0) > b(0)$: $a(t) \to a_\infty > 0$, $b(t) \to 0$,
from rate equations for $d > 2$: $\delta a(t) \sim b(t) \sim e^{-[a(0)-b(0)]\, t}$;
with reaction rate *renormalization* as for $A + A \to \emptyset$:

$$d < 2 : \; \ln b(t) \sim -t^{d/2} \, , \quad d = 2 : \; \ln b(t) \sim -t / \ln(Dt) \; .$$

Add *diffusion*: $\partial_t \, a(x,t) = D_A \, \nabla^2 a(x,t) - \lambda \, a(x,t) \, b(x,t)$;
$D_A = D_B$: *conservation law* $c(t) = a(t) - b(t) = \text{const.}$
$\Longrightarrow$ *diffusive* mode: $\partial_t \, c(x,t) = D \, \nabla^2 c(x,t)$, use Green function

$$G_0(q,t) = \Theta(t) \, e^{-D \, q^2 \, t} \, , \quad G_0(x,t) = \frac{\Theta(t)}{(4\pi \, Dt)^{d/2}} \; e^{-x^2/4Dt} \; ;$$

$$\Longrightarrow \; c(x,t) = \int d^dx' \, G_0(x - x', t) \, c(x', 0) \; .$$

For $a(0) = b(0) \Longrightarrow c(0) = 0$, assume *Poisson distribution*:
$\overline{a(x,0) \, a(x',0)} = a(0)^2 + a(0) \, \delta(x - x')$, $\overline{a(x,0) \, b(x'(0)} = a(0)^2 \Longrightarrow$
$\overline{c(x,0) \, c(x',0)} = 2 \, a(0) \, \delta(x - x')$, average over initial conditions:

$$\overline{c(x,t)^2} = 2 \, a(0) \int d^dx' \, G_0(x - x', t)^2 = 2 \, a(0) \, (8\pi \, Dt)^{-d/2} \; ;$$

since distribution for c will be Gaussian, *local density excess*:

$$\overline{|c(x,t)|} = \sqrt{2 \, \overline{c(x,t)^2} / \pi} = 2 \, \sqrt{a(0)/\pi} \, (8\pi \, Dt)^{-d/4} \; ;$$

$d < d_s = 4$: decays *slower* than $t^{-1} \Longrightarrow$ species *segregation*,
particle distribution *non-uniform*, $a(t) \sim b(t) \sim (Dt)^{-d/4}$,
notice: *mean-field* effect, does *not* require renormalization !

Special initial state in $d = 1$, hard-core particles or $\lambda \to \infty$:
$\ldots ABABABABAB \ldots \longrightarrow \ldots ABABAB \ldots$, as $A + A \to \emptyset$.

Reactions confined to sharp *reaction fronts*:
steady-state approximation, fixed particle *current* J along x:

$$(\partial_x a, \partial_x b) \to \begin{cases} (-J, 0) & \text{as } x \to -\infty \\ (0, J) & \text{as } x \to \infty \end{cases} , \; c(x) = a(x) - b(x) = Jx \, ,$$

$\Longrightarrow D\, \nabla_x^2 a(x) - \lambda\, Jx\, a(x) - \lambda\, a(x)^2 = 0$, solution obeys *scaling*:

$$a(x) = \big(DJ^2/\lambda\big)^{1/3}\; \hat{a}\big((\lambda J/D)^{1/3} x\big) \; ;$$

steady-state reaction front scaling law:

$$R(x) = \lambda\, a(x)\, b(x) = J^{\beta'}\, \hat{R}\big(x\, J^{\alpha'}\big) \; ;$$

$d > 2$: *mean-field* exponents: $\alpha_0' = 1/3$, $\beta_0' = 4/3$;
$d \leq 2$: no anomalous field dimensions, $g_R \to g^*$, since
$[R] = \mu^{d+2}$, $[J] = \mu^{d+1} \Longrightarrow \alpha' = 1/(d+1)$, $\beta' = (d+2)/(d+1)$;
time-dependence: $J(t) \sim a(t)/(Dt)^{1/2} \sim t^{-\lambda}$, $\lambda = (d+2)/4$.

Generalize to *q-species* annihilation $A_i + A_j \to \emptyset$, $1 \leq i < j \leq q$:
no conservation law for $q > 2$; for equal $a_i(0)$, rates D_i, λ_{ij}:
$d \geq 2$: $a_i(t)$ scale as $A + A \to \emptyset$ (obvious for $q = \infty$).
$d = 1$, $\lambda \to \infty$: *topology* allows segregation, *deterministic* model:

$$a_i(t) \sim t^{-\alpha(q)} + C\, t^{-1/2} \; ; \quad \alpha(q) = \frac{q-1}{2\,q} \; ,$$

reaction zone: $J(t) \sim t^{-\lambda(q)}$, $\lambda(q) = (2q-1)/4q$.
In *special* situations recover $A + A \to \emptyset$ behavior, e.g.:
$\ldots ABCDABCDABCD \ldots \longrightarrow \ldots ABADCD \ldots$
Cyclic variant: e.g., $q = 4$, allow only

$$\left.\begin{array}{ll} A + B \to \emptyset\, , & B + C \to \emptyset \\ C + D \to \emptyset\, , & D + A \to \emptyset \end{array}\right\} \begin{array}{l} \text{identify } A = C, B = D \\ \Longrightarrow \; A + B \to \emptyset \text{ scaling.} \end{array}$$

$$2 < d_s(q) = \begin{cases} 4 & q = 2, 4, 6, \ldots \\ 4\cos(\pi/q) & q = 3, 5, 7, \ldots \end{cases} : \; \alpha(q, d) = d/d_s(q) \; ;$$

$d_s(5) = 1 + \sqrt{5}$, $\alpha(5,2) = \frac{1}{2}(\sqrt{5} - 1)$, $\alpha(5,3) = \frac{3}{4}(\sqrt{5} - 1)$.

2.5 Absorbing state transitions, directed percolation

Competing $A \to \emptyset$, $A \rightleftharpoons A + A$ with diffusion:

$$\partial_t \, a(x,t) = -D\left(r - \nabla^2\right) a(x,t) - \lambda \, a(x,t)^2 \ , \quad r = (\kappa - \sigma)/D \ ,$$

biology, ecology: *Fisher–Kolmogoroff* equation (1937).
Continuous transition from active to *absorbing* state at $r = 0$, define *critical exponents* in analogy to equilibrium:

$$\begin{aligned} &\xi(\tau) \sim |\tau|^{-\nu} \ , \quad t_c(\tau) \sim |\tau|^{-z\nu} \ , \quad \tau < 0 \ : \ a_\infty \sim |\tau|^{\beta} \ , \\ &\tau = 0 \ : \ a_c(t) \sim t^{-\alpha} \ , \quad \omega_c(q) \sim |q|^z \ , \quad C(q) \sim |q|^{-2+\eta} \ ; \end{aligned}$$

Gaussian exponent values: $\eta_0 = 0$, $\nu_0 = \frac{1}{2}$, $z = 2$, $\alpha = 1 = \beta$.

Coherent-state path integral (9) for master equation:

$$\begin{aligned} \mathcal{A}[\hat\psi, \psi] = \int d^dx \int dt \Big[&\hat\psi\left(\partial_t - D\,\nabla^2\right)\psi - \kappa\left(1 - \hat\psi\right)\psi \\ &+ \sigma\left(1 - \hat\psi\right)\hat\psi\,\psi - \lambda\left(1 - \hat\psi\right)\hat\psi\,\psi^2\Big] \ ; \end{aligned}$$

shift and rescale $\hat\psi(x,t) = 1 + \sqrt{\frac{\sigma}{\lambda}}\,\widetilde{S}(x,t)$, $\psi(x,t) = \sqrt{\frac{\lambda}{\sigma}}\,S(x,t)$

$$\begin{aligned} \Longrightarrow \quad \mathcal{A}[\widetilde{S}, S] = \int d^dx \int dt \Big\{ &\widetilde{S}\left[\partial_t + D\left(r - \nabla^2\right)\right] S \\ &- u\left(\widetilde{S} - S\right)\widetilde{S}\,S + \lambda\,\widetilde{S}^2\,S^2\Big\} \ , \end{aligned} \tag{10}$$

where $u = \sqrt{\sigma\,\lambda}$, $[u] = \mu^{2-d/2} \Longrightarrow d_c = 4$, $[\lambda] = \mu^{2-d}$ *irrelevant* !
$\lambda = 0$: *Reggeon field theory*, invariant under *rapidity inversion* $S(x,t) \leftrightarrow -\widetilde{S}(x,-t)$; formally *equivalent* to Langevin equation:

$$\begin{aligned} &\frac{\partial\, S(x,t)}{\partial t} = D\left(\nabla^2 - r\right) S(x,t) - u\,S(x,t)^2 + \zeta(x,t) \ , \\ &\langle \zeta(x,t) \rangle = 0 \, , \ \langle \zeta(x,t)\zeta(x',t') \rangle = 2u\,S(x,t)\,\delta(x - x')\delta(t - t') \, , \end{aligned}$$

$\Longrightarrow$ scaling properties of critical *directed percolation* clusters.

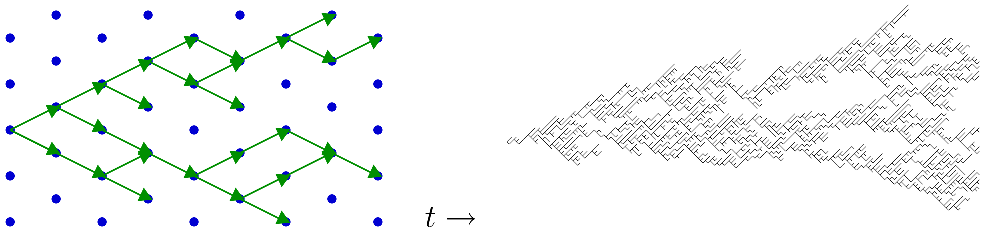

Figure 10: Directed percolation process (left) and critical DP cluster (right).

Phenomenological approach to *simple epidemic process (SEP)*:

1. A *'susceptible' medium* becomes locally *'infected'*, depending on the density n of neighboring 'sick' individuals. The infected regions recover after a brief time interval.

2. The state with $n \equiv 0$ is *absorbing*. This state is equivalent to the *extinction* of the 'disease'.

3. The disease spreads out *diffusively* via the short-range infection (1) of neighboring susceptible regions.

4. Microscopic fast degrees of freedom are incorporated as *local noise* or stochastic forces that respect statement (2), i.e., the noise alone cannot regenerate the disease.

Coarse-grained mesoscopic Langevin representation:

$$\partial_t n = D \left(\nabla^2 - R[n] \right) n + \zeta \ , \quad L[n] = n \, N[n] \ ;$$

near extinction threshold: $R[n] = r + u\, n + \ldots$, $N[n] = v + \ldots$,
higher-order terms *irrelevant* in RG sense, upon rescaling:
response functional (3) yields Reggeon field theory (10) for DP.

$\Longrightarrow$ *DP conjecture* (Janssen 1981, Grassberger 1982):
The critical behavior of an order parameter with *Markovian* stochastic dynamics, decoupled from any other slow variable, that describes a transition from an active to an inactive, *absorbing* state should be in the DP universality class.

2.6 Critical exponents for directed percolation

Explicit evaluation by means of dynamic perturbation theory:

$$\Gamma^{(1,1)}(q,\omega) = i\omega + D(r+q^2) + \frac{u^2}{D}\int_k \frac{1}{i\omega/2D + r + q^2/4 + k^2} \ .$$

Criticality condition, percolation threshold shift $\tau = r - r_c$:

$$\Gamma^{(1,1)}(0,0) = 0 \text{ at } r = r_c \Longrightarrow r_c = -\frac{u^2}{D^2}\int_k \frac{1}{r_c + k^2} + O(u^4) \ ;$$

$$\Gamma^{(1,1)}(q,\omega) = i\omega + D(\tau+q^2) - \frac{u^2}{D}\int_k \frac{i\omega/2D + \tau + q^2/4}{k^2\left(i\omega/2D + \tau + q^2/4 + k^2\right)},$$

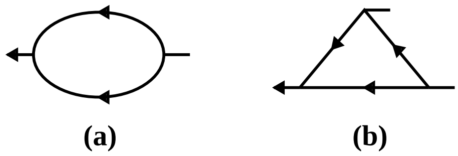

Figure 11: DP renormalization: one-loop diagrams for the vertex functions (a) $\Gamma^{(1,1)}$ (propagator self-energy), and (b) $\Gamma^{(1,2)}$, $\Gamma^{(2,1)}$ (non-linear vertices).

$$\Gamma^{(1,2)}(\{\underline{0}\}) = -\Gamma^{(2,1)}(\{\underline{0}\}) = -2u\left(1 - \frac{2u^2}{D^2}\int_k \frac{1}{(\tau+k^2)^2}\right) .$$

Renormalization: $S_R = Z_S^{1/2}\, S$, $\tilde{S}_R = Z_S^{1/2}\, \tilde{S}$, $D_R = Z_D\, D$,
$\tau_R = Z_\tau\, \tau\, \mu^{-2}$, $u_R = Z_u\, u\, A_d^{1/2} \mu^{(d-4)/2}$, $A_d = \Gamma(3-d/2)/2^{d-1}\pi^{d/2}$,

$$\Longrightarrow \quad Z_S = 1 - \frac{u^2}{2D^2}\frac{A_d\,\mu^{-\epsilon}}{\epsilon} \ , \quad Z_D = 1 + \frac{u^2}{4D^2}\frac{A_d\,\mu^{-\epsilon}}{\epsilon} \ ,$$

$$Z_\tau = 1 - \frac{3u^2}{4D^2}\frac{A_d\,\mu^{-\epsilon}}{\epsilon} \ , \quad Z_u = 1 - \frac{5u^2}{4D^2}\frac{A_d\,\mu^{-\epsilon}}{\epsilon} \ ;$$

$$\Longrightarrow \quad \gamma_S = v_R/2 \ , \quad \gamma_D = -v_R/4 \ , \quad \gamma_\tau = -2 + 3v_R/4 \ ;$$

$$\text{with} \quad v_R = \frac{Z_u^2}{Z_D^2}\frac{u^2}{D^2}A_d\,\mu^{d-4} \ \Longrightarrow\ \beta_v = v_R\left[-\epsilon + 3v_R + O(v_R^2)\right] .$$

Stable RG fixed point for $\epsilon = 4 - d > 0$: $v^* = \epsilon/3 + O(\epsilon^2)$.

Solve Callan–Symanzik equation (5) for *correlation function*:

$$C_R(\tau_R, q, \omega)^{-1} \approx q^2 \, \ell^{\gamma_S^*} \, \hat{C}_R\Big(\tau_R \, \ell^{\gamma_\tau^*}, v^*, \frac{q}{\mu \, \ell}, \frac{\omega}{D_R \, \mu^2 \, \ell^{2+\gamma_D^*}}\Big)^{-1}$$

$\Longrightarrow$ identify *critical exponents* for directed percolation:

$$\eta = -\gamma_S^* = -\frac{\epsilon}{6} + O(\epsilon^2) \ , \quad \frac{1}{\nu} = -\gamma_\tau^* = 2 - \frac{\epsilon}{4} + O(\epsilon^2) \ ,$$
$$z = 2 + \gamma_D^* = 2 - \frac{\epsilon}{12} + O(\epsilon^2) \ ;$$

in the same manner for *order parameter* near v^*:

$$\langle S_R(\tau_R, t) \rangle \approx \mu^{d/2} \, \ell^{(d-\gamma_S^*)/2} \, \hat{S}\big(\tau_R \, \ell^{\gamma_\tau^*}, v_R^*, D_R \, \mu^2 \, \ell^{2+\gamma_D^*} \, t\big) \ \Longrightarrow$$
$$\beta = \frac{\nu(d+\eta)}{2} = 1 - \frac{\epsilon}{6} + O(\epsilon^2) \ , \quad \alpha = \frac{\beta}{z\nu} = 1 - \frac{\epsilon}{4} + O(\epsilon^2) \ .$$

Scaling exponents for various physical quantities are known; beware of *different* definitions for these scaling exponents !

Scaling exponent	$d = 1$	$d = 2$	$d = 4 - \epsilon$
$\xi \sim \lvert\tau\rvert^{-\nu}$	$\nu \approx 1.100$	$\nu \approx 0.735$	$\nu = 1/2 + \epsilon/16 + O(\epsilon^2)$
$t_c \sim \xi^z \sim \lvert\tau\rvert^{-z\nu}$	$z \approx 1.576$	$z \approx 1.73$	$z = 2 - \epsilon/12 + O(\epsilon^2)$
$a_\infty \sim \lvert\tau\rvert^\beta$	$\beta \approx 0.2765$	$\beta \approx 0.584$	$\beta = 1 - \epsilon/6 + O(\epsilon^2)$
$a_c(t) \sim t^{-\alpha}$	$\alpha \approx 0.160$	$\alpha \approx 0.46$	$\alpha = 1 - \epsilon/4 + O(\epsilon^2)$

Table 2: Comparison of the DP critical exponent values from Monte Carlo simulations with the results from the ϵ expansion.

DP critical exponents measured to high precision in Monte Carlo simulations; but *experimental* confirmation ?

Conjecture: *glass transition* in supercooled liquids governed by $T = 0$ DP fixed point (Whitelam, Berthier, Garrahan 2004) ?

2.7 Dynamic percolation, multi-species variants

General epidemic process (GEP) or *epidemic with removal*:

(1') The susceptible medium becomes infected, depending on the densities n *and* m of sick individuals and the *'debris'*, respectively. After a brief time interval, the sick individuals decay into immune debris, which ultimately stops the disease locally by exhausting the supply of susceptible regions.

(2') The states with $n = 0$ and any spatial distribution of m are *absorbing*, and describe the *extinction* of the disease.

Debris: $m(x,t) = \kappa \int_{-\infty}^{t} n(x,t')\, dt' \Longrightarrow$ Langevin description

$$\frac{\partial S(x,t)}{\partial t} = D \left(\nabla^2 - r \right) S(x,t) - D\, u\, S(x,t) \int_{-\infty}^{t} S(x,t')\, dt'$$
$$+\zeta(x,t) \ , \quad \langle \zeta(x,t) \zeta(x',t') \rangle = 2u\, S(x,t)\, \delta(x-x') \delta(t-t') \ ,$$

Quasistatic limit: $\tilde{\varphi}(x) = \widetilde{S}(x, t \to \infty)$, $\varphi(x) = D \int_{-\infty}^{\infty} S(x,t') dt'$

$$\Longrightarrow \mathcal{A}_{\rm qst}[\tilde{\varphi}, \varphi] = \int d^d x \tilde{\varphi} \left[r - \nabla^2 - u \left(\tilde{\varphi} - \varphi \right) \right] \varphi \ ,$$

yields *isotropic percolation* exponents with $d_c = 6$, $\epsilon = 6 - d$: $\eta = -\epsilon/21 + O(\epsilon^2)$, $\nu^{-1} = 2 - 5\epsilon/21 + O(\epsilon^2)$, $\beta = 1 - \epsilon/7 + O(\epsilon^2)$.

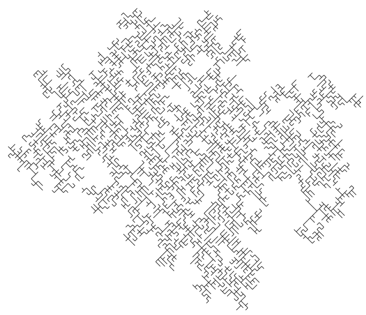

Figure 12: Isotropic percolation cluster.

Dynamics: $z = 2 - \epsilon/6 + O(\epsilon^2) \Longrightarrow$ *dynamic isotropic percolation*.

Multi-species generalizations of directed percolation processes: couple $A_i \to \emptyset$, $A_i \rightleftharpoons A_i + A_i$ via $A_i \rightleftharpoons A_j + A_j (j \neq i)$:

$$\partial_t S_i = D_i\big(\nabla^2 - R_i[S_i]\big)\, S_i + \zeta_i \, , \;\; R_i[S_i] = r_i + \sum_j g_{ij}\, S_j + \ldots \, ,$$
$$\langle \zeta_i \rangle = 0 \, , \;\; \langle \zeta_i(x,t)\, \zeta_j(x',t') \rangle = 2 S_i\, N_i[S_i] \delta(x-x') \delta(t-t')\, \delta_{ij} \, ,$$
$$N_i[S_i] = u_i + \ldots \;\; ;$$

renormalization factors precisely as for single-species DP,
$\Longrightarrow$ *generically* critical behavior in DP universality class,
e.g., extinction threshold in *stochastic Lotka–Volterra* system.

But reactions *generate* $A_i \to A_j$, additional terms $\sum_{j \neq i} g_j\, S_j$,
inter-species couplings become asymptotically *unidirectional*
$\Longrightarrow$ special *multicritical* points when r_i vanish simultaneously;
two species: $A \to \emptyset$, $A \rightleftharpoons A + A$, $B \to \emptyset$, $B \rightleftharpoons B + B$, $A \to B$

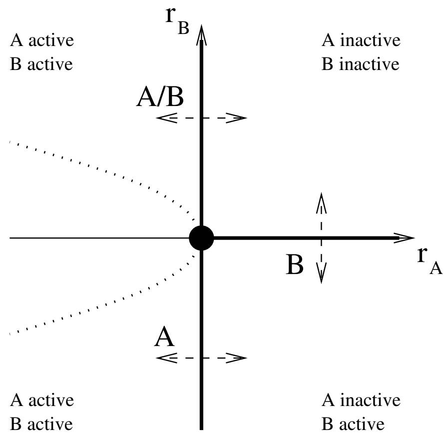

Figure 13: Coupled DP processes: critical lines and multicritical point.

hierarchy of order parameter exponents β_k on the k-th level:

$$\beta_1 = 1 - \frac{\epsilon}{6} + O(\epsilon^2) \, , \; \beta_2 = \frac{1}{2} - \frac{13\epsilon}{96} + O(\epsilon^2) \, , \ldots , \beta_k = \frac{1}{2^k} - O(\epsilon) \;\; ;$$

crossover exponent $\Phi = 1$ to *all* orders.
Analogous features emerge for *multi-species dIP processes.*

2.8 Parity-conserving universality class

Branching/annihilating random walks: $A \to (m+1)\,A$, $k\,A \to \emptyset$
rate equation: $\partial_t\, a(t) = \sigma\, a(t) - k\lambda\, a(t)^k \Longrightarrow$ only active phase:

$$a(t) = \frac{a_\infty}{\left[1 + \left([a_\infty/a(0)]^{k-1} - 1\right) e^{-(k-1)\sigma\, t}\right]^{1/(k-1)}}$$
$$\to a_\infty = (\sigma/k\,\lambda)^{1/(k-1)}\,;$$

$d > d_c(k) = 2/(k-1)$: expect 'transition' at $\sigma_c = 0$,
mean-field exponents: $\eta_0 = 0$, $\nu_0 = \frac{1}{2}$, $\alpha_0 = \frac{1}{k-1} = \beta_0$.
Coherent state path integral (9) for master equation, $k = 2$:

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^dx \int dt \left[\hat{\psi}\left(\partial_t - D\,\nabla^2\right)\psi + \sigma\left(1 - \hat{\psi}^m\right)\hat{\psi}\,\psi \right.$$
$$\left. -\lambda\left(1 - \hat{\psi}^2\right)\psi^2\right]\,,$$

massive propagator $G_0(q,\omega)^{-1} = -i\omega + \sigma + D\,q^2$;
reactions *generate* processes $A \to (m-1)A, (m-3)A, \ldots$;
one-loop: $\gamma_\sigma^m = -2 + \frac{m(m+1)}{2}\,\epsilon \Longrightarrow m = 1, 2$ most relevant.

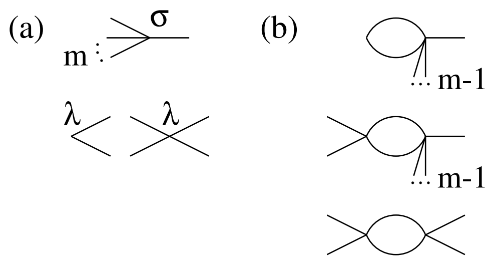

Figure 14: BARW field theory: (a) branching (top) and annihilation (bottom) vertices; (b) one-loop Feynman diagrams generating the $A \to (m-1)A$ process, and renormalizing the branching and annihilation rates, respectively.

$m = 1, 3, \ldots$ *odd*: generates $A \to \emptyset \Longrightarrow$ *DP* universality class,
provided induced decay rate large enough; true for $d \leq d_c = 2$;
non-perturbative 'exact' RG: also in higher dimensions.
$m = 2, 4, \ldots$ *even*: action invariant under $\hat{\psi} \to -\hat{\psi}$, $\psi \to -\psi$,
symmetry reflects particle number *parity conservation (PC)*.

Renormalization: Z factors retain dependence on σ from mass; $\sigma_R = Z_\sigma \sigma / D\,\mu^2$, $\lambda_R = Z_\lambda \lambda\, C_d / D\mu^{2-d}$, $C_d = \Gamma(2-d/2)/2^{d-1}\pi^{d/2}$:

$$\gamma_\sigma = -2 + \frac{3\lambda_R}{(1+\sigma_R)^{2-d/2}}\,, \quad \gamma_\lambda = d - 2 + \frac{\lambda_R}{(1+\sigma_R)^{2-d/2}}\,.$$

$\sigma_R = 0$: pure annihilation fixed point $g^* = 2-d$;
$\sigma_R \to \infty$: *effective* coupling $\Longrightarrow g_R \approx \lambda_R/\sigma_R^{2-d/2}$

$$\beta_g(g_R) \approx g_R \left[\gamma_\lambda - \left(2 - \frac{d}{2}\right)\gamma_\sigma\right] = g_R \left[2 - \frac{10-3d}{2}\, g_R\right];$$

$g^* = 0$: stable, describes (Gaussian) *active* phase,
$g_c = 4/(10-3d) \le 2-d$ if $d \le d_c' = \frac{4}{3}$, unstable: *critical* point.

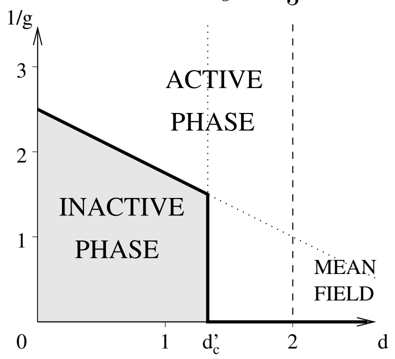

Figure 15: Phase diagram and unstable critical RG fixed point $1/g_c$ for even-offspring BARW (PC universality class) as function of dimension d.

One-loop Callan–Symanzik equation (5) for particle density:
$a(\mu, D, \sigma_R, \lambda_R, t) = (\mu\ell)^d\, \hat{a}\Big(D\mu^2\ell^2 t, \sigma_R \ell^{\gamma_\sigma^*}, \lambda_R \ell^{\gamma_\lambda^*}\Big)$, so $\eta = 0$, $z = 2$;
$d_c' < d = 2-\epsilon \le 2$: $\nu^{-1} = -\gamma_\sigma^* = 2-3\epsilon$, $\alpha = d/2$, $\beta = d\nu = z\nu\alpha$.
At *fixed* $d \le d_c' = \frac{4}{3}$, using the mean-field expression:

$$a(\mu, D, \sigma_R, \lambda_R, t) = \mu^d\, \frac{\sigma_R}{\lambda_R}\, \ell^{d+\gamma_\sigma^*-\gamma_\lambda^*}\, \tilde{a}\Big(\sigma_R \mu^2 t\, \ell^{2+\gamma_\sigma^*}, |g_c - g_R|\, \ell^{\beta_g'^*}\Big)\,,$$

$$\Longrightarrow \quad \nu = \frac{2+\gamma_\sigma^*}{-\beta_g'^*} = \frac{3}{10-3d}\,, \quad \beta = \frac{d+\gamma_\sigma^*-\gamma_\lambda^*}{-\beta_g'^*} = \frac{4}{10-3d}\,.$$

q species: $A_i \to 3A_i$ (rate σ), $A_i \to A_i + 2A_{j\neq i}(\sigma')$, $A_i + A_i \to \emptyset$;
$q \ge 2$: σ irrelevant, $\sigma_c' = 0$, and $\nu = 1/d$, $z = 2$, $\alpha = d/2$, $\beta = 1$.

2.9 Some open issues

Reaction–diffusion systems:

- better analytical understanding of PC universality class, non-perturbative methods ?
- higher-order reactions; example: *pair contact process with diffusion (PCPD)*, $A + A \to \emptyset / A$, $A + A \to (n+2)\,A$: inactive state as for PC, $a \to \infty$ in active phase $\Longrightarrow$ well-defined transition requires occupation number *restrictions*; in field theory: branching process $\to \sigma\,(1 - \hat{\psi})\,\hat{\psi}\,\psi\,e^{-2v\,\hat{\psi}\,\psi}$, $[v] = \kappa^{-d} \to$ expand in v; resulting theory has *no* stable finite RG fixed point $\Longrightarrow$ continuum limit clearly subtle; requires explicit inclusion of particle *pair* density ?
- generalization to full classification of scaling properties in *multi-species* systems incomplete and formidable program;
- *experimental* realizations, confirmations, applications ?
  $A + A \to A$: exciton kinetics in quasi-1D polymers; others ?
- role of *quenched disorder* quite poorly understood ... ?
  DP, random percolation threshold $\Longrightarrow$ RG runaway flows.

Generally for non-equilibrium systems:

- role of universality less prominent, but still *useful* concept; complete *classification* possible ?
- connection to *'real'* systems, *experimental* confirmation ?
- applications to biological systems fruitful and promising, but *non-universal* features often crucial.